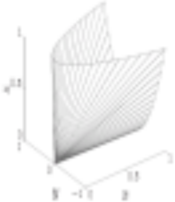
Positive semidefinite cone: **Primal Description**

Consider a positive semi-definite matrix S in $\Re^2$. Then S must be of the form

$$S = \begin{bmatrix} \mathbf{x} & y \\ y & z \end{bmatrix} \tag{33}$$

We can represent the space of matrices $\mathcal{S}_+^2$ of the form $S \in \mathcal{S}_+^2$ as a three dimensional space with non-negative $\mathbf{x}$, y and z coordinates and a non-negative determinant. This space



corresponds to a cone as shown in the Figure above.

Positive semidefinite cone: **Dual Description**

Instead of all vectors $\mathbf{z} \in \Re^n$, we can, without loss of generality, only require the above inequality to hold for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1$.

1. $S_+^n = \{A \in S^n | A \succeq 0\} = \{A \in S^n | \mathbf{y}^T A \mathbf{y} \succeq 0, \forall \|\mathbf{y}\|_2 = 1\}$
2. So, $S_+^n = \cap_{\|\mathbf{y}\|=1} \{A \in S| < \mathbf{y}^T\mathbf{y}, A > \succeq 0\}$
3. $\mathbf{y}^T A \mathbf{y} = \sum_i \sum_j y_i a_{ij} y_j = \sum_i \sum_j (y_i y_j) a_{ij} = < \mathbf{y}\mathbf{y}^T, A > = tr((\mathbf{y}\mathbf{y}^T)^T A) = tr(\mathbf{y}\mathbf{y}^T A)$
   - One parametrization for $\mathbf{y}$ such that $\|\mathbf{y}\|_2 = 1$ is

Frobenius inner product

$$\mathbf{y} = \begin{bmatrix} Cos(\theta) \\ Sin(\theta) \end{bmatrix} \quad (34)$$

$$\mathbf{y}\mathbf{y}^T = \begin{bmatrix} Cos^2(\theta) & Cos(\theta)Sin(\theta) \\ Cos(\theta)Sin(\theta) & Sin^2(\theta) \end{bmatrix} \quad (35)$$

   - Assignment 1: Plot a finite # of halfspaces parameterized by (θ).

Positive semidefinite cone: **Dual Description**

1. S_+^n = intersection of infinite # of half spaces belonging to $R^{n(n+1)/2}$ [Dual Representation]
    1. Cone boundary consists of all singular p.s.d. matrices having at-least one 0 eigenvalue.
    2. Origin = O = matrix with all 0 eigenvalues.
    3. Interior consists of all full rank matrices A (rank $A = m$) i.e. $A \succ 0$.

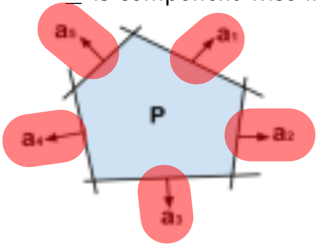
HW: $N = \|.\|_\infty$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})} = \sup_{\|\mathbf{x}\|=1} N(A\mathbf{x})$

1. If $N(\mathbf{x}) = \max_i |x_i|$ then $N(A\mathbf{x}) = \max_i |\sum_{j=1}^{m} a_{ij}x_j| \leq \max_i \sum_{j=1}^{m} |a_{ij}||x_j| \leq\leq \max_i \sum_{j=1}^{m} |a_{ij}|$ where the last inequality is attained by considering a $\mathbf{x} = [1, 1..1, 1...1]$ which has 1 in all positions. Then $\|\mathbf{x}\|_\infty = 1$ and for such an $\mathbf{x}$, the upper bounded on the supremum in indeed attained.
2. Therefore, it must be that $\|A\mathbf{x}\|_1 = \max_i \sum_{j=1}^{m} |a_{ij}|$ (the maximum absolute row sum)
3. That is,

$$M_N(A) = \|A\mathbf{x}\|_1 = \max_i \sum_{j=1}^{m} |a_{ij}|$$

Convex Polyhedron

- Solution set of finitely many inequalities or equalities: $A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}$
  - $A \in \Re^{m \times n}$
  - $C \in \Re^{p \times n}$
  - $\preceq$ is component wise inequality



- This is a **Dual or *H* Description**: Intersection of finite number of half-spaces and hyperplanes.
- **Primal or *V* Description**: Can you define convex polyhedra in terms of convex hull?
  1. Convex hull of finite # of points ⇒ **Convex Polytope**
  2. Conic hull of finite # of points ⇒ **Polyhedral Cone**
  3. Convex hull of $n+1$ affinely independent points ⇒ **Simplex**

V = vertices or points

Convex combinations and Convex Hull

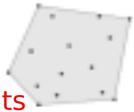
- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\})$$

$$\text{with} \quad \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0.$$

- Equivalent Definition of Convex Set: *C* is convex iff it is closed under generalized convex combinations.
- **Convex hull or** *conv*(S) is the set of all convex combinations of points in the set *S*.
- *conv*(S) = The smallest convex set that contains *S*. *S* may not be convex but *conv*(S) is.
  - Prove by contradiction that if a point lies in another smallest convex set , and not in *conv*(S), then it must be in *conv*(S).

S is a finite set of points



Original set was not convex, whereas the convex hull is

- The idea of convex combination can be generalized to include infinite sums, integrals, and, in most general form, probability distributions.

Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Balls in $\Re^n$]: Consider a point $\mathbf{x} \in \Re^n$. Then the closed ball around $\mathbf{x}$ of radius ϵ is

$$\mathcal{B}[\mathbf{x}, \epsilon] = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| \leq \epsilon\right\}$$

Likewise, the open ball around $\mathbf{x}$ of radius ϵ is defined as

$$\mathcal{B}(\mathbf{x}, \epsilon) = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| < \epsilon\right\}$$

For the 1-D case, open and closed balls degenerate to open and closed intervals respectively.

Definition

[Boundedness in $\Re^n$]: We say that a set $\mathcal{S} \subset \Re^n$ is *bounded* when there exists an $\epsilon > 0$ such that $\mathcal{S} \subseteq \mathcal{B}[0, \epsilon]$.

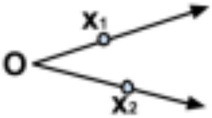
In other words, a set $\mathcal{S} \subseteq \Re^n$ is bounded means that there exists a number $\epsilon > 0$ such that for all $\mathbf{x} \in \mathcal{S}$, $||\mathbf{x}|| \leq \epsilon$.

Convex Polytope: **Primal** and **Dual** Descriptions

Dual or H Description: A Convex Polytope P is a **Bounded Convex Polyhedron**. That is, is solution set of finitely many inequalities or equalities: $P = \{\mathbf{x} | A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}\}$ where $A \in \Re^{m \times n}$, $C \in \Re^{p \times n}$ such that P is also bounded.

Primal or V Description : If $\exists\ S \subset$ P s.t. $|S|$ is finite and $P = conv(S)$, then P is a **Convex Polytope**.

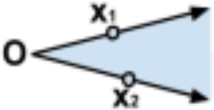
Conic combinations and Conic Hull



- Recap **Cone**: A set C is a cone if $\forall \mathbf{x} \in C$, $\theta \mathbf{x} \in C$ for $\theta \geq 0$.
- **Conic (nonnegative) Combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1 \mathbf{x}_1 + \theta_2 \mathbf{x}_2 + ... + \theta_k \mathbf{x}_k$$
$$\text{with} \quad \theta_i \geq 0.$$

- **Conic hull or** *conic*(S)**:** The set that contains all conic combinations of points in set S.



- *conic*(S) = Smallest conic set that contains S.

Polyhedral Cone: **Primal** and **Dual** Descriptions

Dual or *H* Description : A Polyhedral Cone P is a Convex Polyhedron with $\mathbf{b} = 0$. That is, $\{\mathbf{x} | A\mathbf{x} \succeq 0\}$ where $A \in \Re^{m \times n}$ and $\succeq$ is component wise inequality.

Primal of *V* Description : If $\exists\ S \subset$ P s.t. $|S|$ is finite and $P = cone(S)$, then P is a **Polyhedral Cone**.

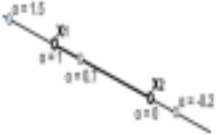
Affine combinations, Affine hull and Dimension of a set S

- **Affine Combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k$$

$$\text{with} \quad \sum_i \theta_i = 1$$

- **Affine hull or aff(S):** The set that contains all affine combinations of points in set S = Smallest affine set that contains S.



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- Dimension of a set S = dimension of $aff(S)$ = dimension of vector space V such that $aff(S) = \mathbf{v} + V$ for some $\mathbf{v} \in aff(S)$.
- $S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$ is set of $n+1$ *affinely independent* points if $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_{n+1} - \mathbf{x}_0\}$ are linearly independent.

Simplex (plural: simplexes) Polytope: **Primal** and **Dual** Descriptions

Dual or *H* Description: An n Simplex S is a convex Polytope with of affine dimension n and having $n+1$ corners. That is, is solution set of finitely many inequalities or equalities: $S = \{\mathbf{x} | A\mathbf{x} \preceq \mathbf{b} \ , \ C\mathbf{x} = \mathbf{d}\}$ where $A \in \Re^{m \times n}$, $C \in \Re^{p \times n}$ such that S with affine dimension n and having $n+1$ corners.

Primal or *V* Description: Convex hull of $n+1$ affinely independent points. Specifically, let $S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$ be a set of $n+1$ affinely independent points, then an n-dimensional simplex is $conv(S)$.
Simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.

Convexity preserving operations

Assigment 1 is about ascertaining convexity or non-convexity and other properties of a whole bunch of sets

In practice if you want to establish the convexity of a set $\mathcal{C}$, you could either

1. prove it from first principles, *i.e.*, using the definition of convexity or
2. prove that $\mathcal{C}$ can be built from simpler convex sets through some basic operations which preserve convexity.

Some of the important operations that preserve complexity are:

1. Intersection
2. Affine Transform
3. Perspective and Linear Fractional Function

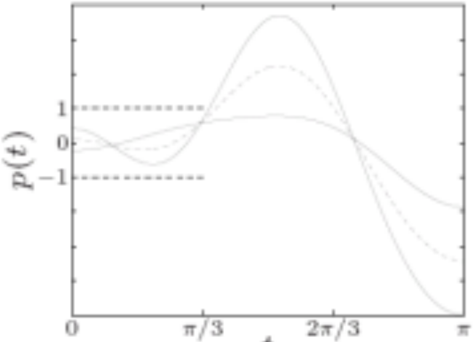
Closure under Intersection

The intersection of any number of convex sets is convex. Consider the set $\mathcal{S}$:

$$\mathcal{S} = \left\{ \mathbf{x} \in \Re^n \mid |p(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3} \right\} \quad (36)$$

where

$$p(t) = x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \quad (37)$$



p(t) plotted for different values of x1..xm

Closure under Intersection (contd.)

Any value of t that satisfies $|p(t)| \leq 1$, defines two regions, *viz.*,

$$\Re^{\leq}(t) = \left\{\mathbf{x} \mid x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \leq 1\right\}$$

and

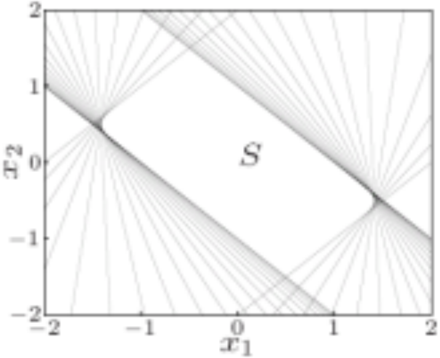
$$\Re^{\geq}(t) = \left\{\mathbf{x} \mid x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \geq -1\right\}$$

Each of the these regions is convex and for a given value of t, the set of points that may lie in $\mathcal{S}$ is given by $\Re(t) = \Re^{\leq}(t) \cap \Re^{\geq}(t)$

Closure under Intersection (contd.)

$\Re(t)$ is also convex. However, not all the points in $\Re(t)$ lie in $\mathcal{S}$, since the points that lie in $\mathcal{S}$ satisfy the inequalities for every value of t. Thus, $\mathcal{S}$ can be given as:

$$\mathcal{S} = \cap_{|t| \leq \frac{\pi}{3}} \Re(t)$$



Closure under Affine transform

An affine transform is one that preserves

- Collinearity between points, *i.e.*, three points which lie on a line continue to be collinear after the transformation.
- Ratios of distances along a line, *i.e.*, for distinct colinear points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, $\frac{||\mathbf{p}_2 - \mathbf{p}_1||}{||\mathbf{p}_3 - \mathbf{p}_2||}$ is preserved.

Verify that affine transformation preserves these..

An affine transformation or affine map between two vector spaces $f : \Re^n \rightarrow \Re^m$ consists of a linear transformation followed by a translation:

$$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$$

where $A \in \Re^{n \times m}$ and $\mathbf{b} \in \Re^m$.

Closure under Affine transform (contd.)

In the finite-dimensional case each affine transformation is given by a matrix A and a vector $\mathbf{b}$. The image and pre-image of convex sets under an affine transformation defined as

$$f(\mathbf{x}) = \sum_{i}^{n} x_i a_i + b$$

yield convex sets[8]. Here a_i is the i^{th} row of A. The following are examples of convex sets that are either images or inverse images of convex sets under affine transformations:

1. the solution set of linear matrix inequality $(A_i, B \in \mathcal{S}^m)$

   $$\left\{\mathbf{x} \in \Re^n \mid x_1 A_1 + \ldots + x_n A_n \preceq B\right\}$$

   is a convex set. Here $A \preceq B$ means $B - A$ is positive semi-definite[9]. This set is the inverse image under an affine mapping of the positive semidefinite cone

[8] Exercise: Prove.

[9] The inequality induced by positive semi-definiteness corresponds to a generalized inequality $\preceq_K$ with $K = \mathcal{S}^n_+$.

Closure under Affine transform (contd.) H/W: Prove

In the finite-dimensional case each affine transformation is given by a matrix A and a vector $\mathbf{b}$. The image and pre-image of convex sets under an affine transformation defined as

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is a convex set. Here $A \preceq B$ means $B - A$ is positive semi-definite[9]. This set is the inverse image under an affine mapping of the spositive semi-definite cone. That is, $f^{-1}(cone) = \left\{\mathbf{x} \in \Re^n \mid B - (x_1 A_1 + \ldots + x_n A_n) \in \mathcal{S}_+^m\right\} = \left\{\mathbf{x} \in \Re^n \mid B \geq (x_1 A_1 + \ldots + x_n A_n)\right\}$.

[8] Exercise: Prove.

[9] The inequality induced by positive semi-definiteness corresponds to a generalized inequality $\preceq_K$ with $K = \mathcal{S}_+^n$.

Closure under Affine transform (contd.)

2. hyperbolic cone ($P \in \mathcal{S}_+^n$), which is the inverse image of the

Closure under Affine transform (contd.)

2. hyperbolic cone $\left(P \in \mathcal{S}_+^n\right)$, which is the inverse image of the norm cone $\mathcal{C}_{m+1} = \left\{(\mathbf{z}, u) | ||\mathbf{z}|| \leq u, u \geq 0, \mathbf{z} \in \Re^m\right\} = \left\{(\mathbf{z}, u) | \mathbf{z}^T\mathbf{z} - u^2 \leq 0,\ u \geq 0, \mathbf{z} \in \Re^m\right\}$ is a convex set. The inverse image is given by $f^{-1}\left(\mathcal{C}_{m+1}\right) = \left\{\mathbf{x} \in \Re^n \mid \left(A\mathbf{x}, \mathbf{c}^T\mathbf{x}\right) \in \mathcal{C}_{m+1}\right\} = \left\{\mathbf{x} \in \Re^n | \mathbf{x}^T A^T A\mathbf{x} - (\mathbf{c}^T\mathbf{x})^2 \leq 0\right\}$. Setting, $P = A^T A$, we get the equation of the hyperbolic cone:

$$\left\{\mathbf{x} \mid \mathbf{x}^T P\mathbf{x} \leq (\mathbf{c}^T\mathbf{x})^2, \mathbf{c}^T\mathbf{x} \geq 0\right\}$$

Closure under Perspective and linear-fractional functions

H/w: Prove

The perspective function $P : \Re^{n+1} \rightarrow \Re^n$ is defined as follows:

$$\begin{aligned} &P : \Re^{n+1} \rightarrow \Re^n \text{ such that} \\ &P(x,t) = x/t \qquad \textbf{dom } P = \{(x,t) \mid t > 0\} \end{aligned} \tag{38}$$

Shrinks from n+1 to n by division using t

The linear-fractional function f is a generalization of the perspective function and is defined as: $\Re^n \rightarrow \Re^m$:

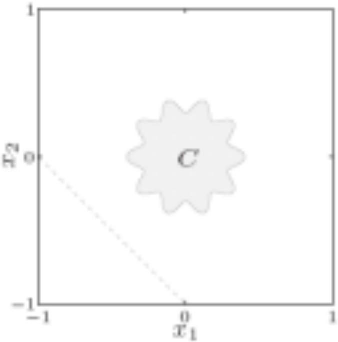
$$\begin{aligned} &f : \Re^n \rightarrow \Re^m \text{ such that} \\ &f(\mathbf{x}) = \frac{A\mathbf{x}+\mathbf{b}}{\mathbf{c}^T\mathbf{x}+d} \qquad \textbf{dom } f = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} + d > 0\} \end{aligned} \tag{39}$$

The images and inverse images of convex sets under perspective and linear-fractional functions are convex[10].

[10] Exercise: Prove.

Closure under Perspective and linear-fractional functions (contd)

The Figure below shows an example set.



Closure under Perspective and linear-fractional functions (contd)

Consider the linear-fractional function $f = \frac{1}{x_1+x_2+1}x$. The following Figure shows the image of the set (from the prevous slide) under the linear-fractional function f.