

Let us resume our discussion on linear programs (LP), dual of LP, conic programs & their duals

[Ref page 5 of <http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf>]

LP Affine objective

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & -Ax + b \leq 0 \end{aligned}$$

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & -Ax + b \leq_K 0 \end{aligned}$$

Conic Program (CP)

Let: $\lambda \geq 0$ (i.e. $\lambda \in \mathbb{R}_+$)

then $\lambda^T (-Ax + b) \leq 0$

$$\begin{aligned} \Rightarrow c^T x & \geq c^T x + \lambda^T (-Ax + b) \\ & = \lambda^T b + (c - A^T \lambda)^T x \end{aligned}$$

$$\geq \min_x \lambda^T b + (c - A^T \lambda)^T x$$

$$= \begin{cases} \lambda^T b & \text{if } A^T \lambda = c \\ -\infty & \text{if } A^T \lambda \neq c \end{cases}$$

independent of x

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \end{aligned} \geq \begin{aligned} \max_{\lambda \geq 0} \quad & b^T \lambda \\ \text{s.t.} \quad & A^T \lambda = c \end{aligned}$$

Primal LP (lower bounded)

Dual LP (upper bounded)

K is a regular / proper cone
Generalised cone program

$$\begin{aligned} \min_{x \in S} \quad & \langle c, x \rangle_S \\ \text{subject to} \quad & Ax - b \in K \end{aligned}$$

We need an equivalent $\lambda \in K^*$ s.t.

$$\langle \lambda, Ax - b \rangle \geq 0$$

This K^* s.t.

$$K^* = \left\{ \lambda \mid \langle \lambda, Ax - b \rangle \geq 0 \quad \forall Ax - b \in K \right\}$$

is called the DUAL CONE of K (i.e. elements that have the inner prod with each element of K)

by dual)

by primal)

Called the weak duality theorem for Linear Program

$K_* = \{\lambda : \lambda^T \xi \geq 0 \forall \xi \in K\}$ is the cone dual to K

{defn on page 7 of

<http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf>

With this, prove the following weak duality theorem for CONIC PROGRAM

$\min_{x \in S} \langle c, x \rangle$
s.t. $Ax \succeq_K b$

Primal CP (lower bounded by dual)

$\max_{\lambda \in K^*} \langle b, \lambda \rangle$
s.t. $A^T \lambda = c$

Dual CP (upper bounded by primal)

Note: Both LP & CP dealt

with affine objective

CP dealt with the generalised conic inequalities

Later, in convex programs, we will deal with the more general convex functions in the objective

Notes:

1) If $K = \mathbb{R}_+^n$, the CP is an LP

If $K = S_+^n$, the CP is an SDP

Set of all $n \times n$ symmetric positive semi-definite matrices

2) Any generic convex program can be expressed as a cone program (CP) [H/W]

HOW ABOUT STRONG DUALITY FOR LPs?

[Page 21 of http://www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf]

Theorem 1.2.2 [Duality Theorem in Linear Programming] Consider a linear programming program

$$\min_x \left\{ c^T x \mid Ax \geq b \right\} \quad (\text{LP})$$

along with its dual

$$\max_y \left\{ b^T y \mid A^T y = c, y \geq 0 \right\} \quad (\text{LP}^*)$$

Then

- 1) The duality is symmetric: the problem dual to dual is equivalent to the primal;
- 2) The value of the dual objective at every dual feasible solution is \leq the value of the primal objective at every primal feasible solution

3) The following 5 properties are equivalent to each other:

- (i) The primal is feasible and bounded below.
- (ii) The dual is feasible and bounded above.
- (iii) The primal is solvable.
- (iv) The dual is solvable.
- (v) Both primal and dual are feasible.



Whenever (i) \equiv (ii) \equiv (iii) \equiv (iv) \equiv (v) is the case, the optimal values of the primal and the dual problems are equal to each other. : Strong duality = (2) + (3)

Proof of (i) from page 21 of

http://www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf

Proof. 1) is quite straightforward: writing the dual problem (LP*) in our standard form, we get

$$\min_y \left\{ -b^T y \mid \begin{bmatrix} I_m \\ A^T \\ -A^T \end{bmatrix} y - \begin{pmatrix} 0 \\ -c \\ c \end{pmatrix} \geq 0 \right\},$$

where I_m is the m -dimensional unit matrix. Applying the duality transformation to the latter problem, we come to the problem

$$\max_{\xi, \eta, \zeta} \left\{ \begin{array}{l} 0^T \xi + c^T \eta + (-c)^T \zeta \\ \xi \geq 0 \\ \eta \geq 0 \\ \zeta \geq 0 \\ \xi - A\eta + A\zeta = -b \end{array} \right\},$$

which is clearly equivalent to (LP) (set $x = \eta - \zeta$).

Proof for (i) \equiv (ii) \equiv (iii) \equiv (iv) \equiv (v).

Let us prove that (i) \Rightarrow (iv)

Let δ be a lower bound on

the primal. That is,

$$\forall x \text{ s.t. } Ax \geq b, \\ c^T x \geq \delta$$

We will first prove that:

\Downarrow for small $\alpha > 0$

$$\forall x \text{ s.t. } Ax \geq b \\ c^T x > \delta - \alpha \quad \rightarrow \textcircled{0}$$

Recall: $Ax \geq b$ has a soln iff $y^T b \geq 0$
 $\textcircled{1}$ for all y s.t. $y^T A = 0$ [can be proved by analysing Gauss elimination]

called Theorem of Alternate

(results in Farkas Lemma)

Exactly one of the following statements are true:

$\textcircled{2}$ (1) $\exists x \in \mathbb{R}^n$ s.t. $Ax \geq b, x \geq 0$
(2) $\exists y \in \mathbb{R}^m$ s.t. $y^T A \geq 0, y^T b < 0$

Independent proof of Farkas' lemma:

(a) NOT BOTH Suppose x satisfies ① & y satisfies ②. Then $0 > y^T b \geq y^T A x \geq 0$
 \Downarrow \Downarrow
 $\therefore Ax = b$ $\therefore y^T A x > 0$
& $x \geq 0$
 \Rightarrow CONTRADICTION

(b) At least one Suppose ① is infeasible
we will show that ② is feasible

(i) Consider the closed convex set

$$S = \{ Ax : x \geq 0 \}$$

Then $b \notin S$ since ① is infeasible

(ii) According to the separating hyperplane theorem, \exists a hyperplane separating S from $\{b\}$. That is, $\exists y \in \mathbb{R}^m$ & $l \in \mathbb{R}$ s.t. $y^T b < l$ & $y^T s \geq l \quad \forall s \in S$

(iii) Since $0 \in S$, $l \leq 0 \Rightarrow y^T b < 0$

(iv) $s = Ax$ for some $x \geq 0$
 $\therefore y^T A x \geq l \quad \forall x \geq 0 \Rightarrow y^T A \geq 0$

LHS should hold for arbitrarily large $x \geq 0$

Farkas' lemma
corollary

③

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, exactly one of the following two systems holds:

① $\exists x \in \mathbb{R}^n$ s.t. $Ax \geq b$.

② $\exists y \in \mathbb{R}^m$ s.t. $y^T A = 0$ & $y^T b > 0$

Proof: Apply Farkas' lemma to:

① $\exists x \in \mathbb{R}^n, s \in \mathbb{R}^m$ s.t.

$$Ax - Is = b, \quad x, s \geq 0$$

② n/w: Fill it up

We know that

whenever $Ax \geq b, c^T x > l$
ie the system below has
no soln:

$$-c^T x \geq -l, Ax \geq b$$

$$\textcircled{4} \exists y \text{ s.t. } A^T y = c \text{ \& } b^T y \geq l$$

Apply Farkas' corollary $\textcircled{3}$ & formulate 2 problems,
exactly one of which has a solution

$$\textcircled{1} \exists x \in \mathbb{R}^n \text{ s.t. } Ax \geq b, -c^T x > -l$$

The strict lower bound
on primal as per
(i)

$$\textcircled{2} \exists y \in \mathbb{R}^m \text{ \& } \lambda \in \mathbb{R} \text{ s.t. } \begin{bmatrix} y^T & \lambda \end{bmatrix} \begin{bmatrix} A \\ -c^T \end{bmatrix} = 0$$

$$\begin{bmatrix} y^T & \lambda \end{bmatrix} \begin{bmatrix} b \\ -l \end{bmatrix} > 0$$

$$\text{\& } \begin{pmatrix} y \\ \lambda \end{pmatrix} \geq 0$$

from $\textcircled{0}$ we know that l is a lower bound on
 $\{c^T x \mid Ax \geq b\}$

\Rightarrow ① above is infeasible. \therefore By Farkas' Corollary,

② is feasible. Two possibilities exist

① $\lambda \neq 0$. Then $z = \frac{y^T}{\lambda}$ & $\frac{\lambda}{\lambda} = 1$ is also

a solution to ②
 $\therefore (z^T, 1)$ is a soln to ②
 $\Rightarrow z^T A = c^T$ & $z^T b > l$

That is, if l is a **strict** lower bound on the primal, $\exists z \in \mathbb{R}^m$ that is dual feasible (i.e. $z^T A = c^T$) and such that dual objective at z ($z^T b$) is strictly greater than l

② $\lambda = 0 \Rightarrow y^T A = 0$ & $y^T b > 0$. \rightarrow say $= \alpha$

$\therefore \exists z \geq 0$ & $\alpha > 0$ s.t.

$$z = \frac{\alpha y}{\alpha}, \quad z^T A = 0 \quad \& \quad z^T b = l$$

\therefore for $\epsilon > 0$

$$A^T(z + \epsilon y) = 0$$

$$z + \epsilon y \geq 0 \quad (\because y \geq 0)$$

$$b^T(z + \epsilon y) = l + \epsilon b^T y > l$$

Again this means that if l is a **strict** lower bound on the primal, $\exists z + \epsilon y$ that is dual feasible & st $b^T(z + \epsilon y) > l$

Thus if $l + \alpha$ is a lower bound on primal,
 \exists a dual feasible y st $b^T y > l + \alpha > 0$

\therefore By the weak duality theorem, $\exists l + \alpha$ st

$$\text{OPT}(\text{LP}) = l + \alpha = \text{OPT}(\text{LP}^*)$$

Theorem 1.2.3 [Necessary and sufficient optimality conditions in linear programming] Consider an LP program (LP) along with its dual (LP*). A pair (x, y) of primal and dual feasible solutions is comprised of optimal solutions to the respective problems if and only if

$$y_i [Ax - b]_i = 0, \quad i = 1, \dots, m,$$

[complementary slackness]

likewise as if and only if

$$c^T x - b^T y = 0$$

[zero duality gap]

special case of Karush Kuhn Tucker (KKT) conditions to be discussed later

Proof sketch: [H/W Complete the proof rigorously]

only if From Theorem 1.2.2, if x & y are pts of optimal primal & dual solns respectively, then

$$A^T y = c \Rightarrow (A^T y)^T x = c^T x = y^T b \Rightarrow y^T (Ax - b) = 0$$

(and since $y \geq 0$
 $Ax - b \geq 0$) $\Rightarrow \forall i, y_i [Ax - b]_i = 0$

if $y_i [Ax - b]_i = 0 \Rightarrow y^T (Ax - b) = 0 \Rightarrow y^T b = y^T Ax$
 \Rightarrow Dual is solvable (condition 3.(iv) of Theorem 1.2.2)

can be proved independently

\Rightarrow conditions (1) & (3) of Theorem 1.2.2 are met

Similar Duality theorem for CP:

[page 7 of

<http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf>]

Note $\text{dual}(\text{dual}(K)) = K$ if K is a closed dual(K) is ALWAYS a closed cone

$$\min_x \left\{ c^T x : \underbrace{Ax - b \geq_K 0}_{\Leftrightarrow Ax - b \in K} \right\}$$

(CP)

$$\max_{\lambda} \{ b^T \lambda : A^T \lambda = c, \lambda \geq_{K^*} 0 \},$$

(D)

Theorem 2.1. Assuming A in (CP) is of full column rank, the following is true:

(i) The duality is symmetric: (D) is a conic problem, and the conic dual to (D) is (equivalent to) (CP);

(ii) [weak duality] $\text{Opt}(D) \leq \text{Opt}(CP)$;

→ Already proved

(iii) [strong duality] If one of the programs (CP), (D) is bounded and strictly feasible (i.e., the corresponding affine plane intersects the interior of the associated cone), then the other is solvable and $\text{Opt}(CP) = \text{Opt}(D)$. If both (CP), (D) are strictly feasible, then both programs are solvable and $\text{Opt}(CP) = \text{Opt}(D)$;

(iv) [optimality conditions] Assume that both (CP), (D) are strictly feasible. Then a pair (x, λ) of feasible solutions to the problem is comprised of optimal solutions iff $c^T x = b^T \lambda$ ("zero duality gap"), same as iff $\lambda^T [Ax - b] = 0$ ("complementary slackness").

H/W: Prove these in a manner similar to the duality theorem for LP

Note: Duality theorems for LP and CP are special cases of Lagrange duality that we will discuss later in the course

Dual cones and generalized inequalities

dual cone of a cone K :

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbf{R}_+^n$: $K^* = \mathbf{R}_+^n$
- $K = \mathbf{S}_+^n$: $K^* = \mathbf{S}_+^n$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual cones**

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0$$

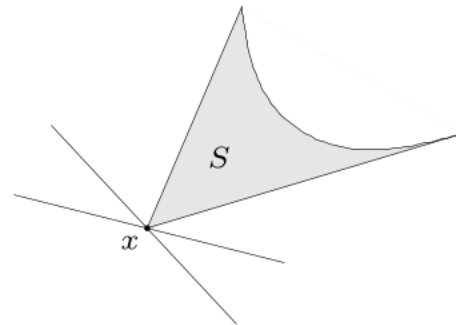
Convex sets

if $\frac{1}{p} + \frac{1}{q} = 1$ then $\|x\|_p$ is called the dual norm of $\|x\|_q$

Minimum and minimal elements via dual inequalities

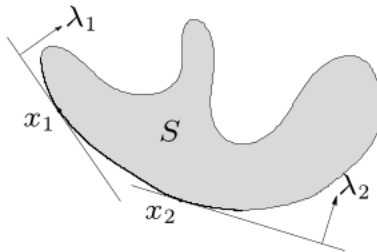
minimum element w.r.t. \preceq_K

x is minimum element of S iff for all $\lambda \succeq_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S



minimal element w.r.t. \preceq_K

- if x minimizes $\lambda^T z$ over S for some $\lambda \succeq_{K^*} 0$, then x is minimal



- if x is a minimal element of a convex set S , then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

Thus (continuing our story of dual descriptions of sets)

3) if $C = \text{Conic set} \subseteq S$ and B is its basis

$$C = \text{conic hull}(B) = \left\{ s \in S \mid \langle s, b \rangle_S \geq 0 \forall b \in B^* \right\}$$

where B^* is basis for $C^* = \left\{ s^* \in S \mid \langle s^*, c \rangle_S \geq 0 \forall c \in C \right\}$

Some more facts on dual cones:

(a) Dual cone $K^* = \left\{ y \mid \langle x, y \rangle \geq 0 \forall x \in K \right\}$
is convex, even if K is not

i.e., a cone

i.e., a cone

Proof: (i) $0 \in K^*$ since $\langle x, 0 \rangle = 0$

(ii) if $y_1 \in K^*$ & $y_2 \in K^*$ and $\theta_1, \theta_2 \geq 0$

↓
Reals

$$\begin{aligned} \text{then } \langle x, \theta_1 y_1 + \theta_2 y_2 \rangle &= \theta_1 \langle x, y_1 \rangle + \theta_2 \langle x, y_2 \rangle \\ &= \theta_1 \langle x, y_1 \rangle + \theta_2 \langle x, y_2 \rangle \geq 0 \end{aligned}$$

In fact K^* is a closed convex cone

(b) if $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$

② Interior(K^*) = $\{y \mid y^T x > 0 \ \forall \ x \in K\}$

① If interior(K) $\neq \emptyset$, K^* is pointed

③ K^{**} is the closure of K (\because if K is closed, $K^{**} = K$). Also if K is a proper cone, so is its dual

④ If closure of K is pointed then K^* has non empty interior
 [Refer Boyd's book, exercise 2.31 for ①-④]

⑧ If $V \subseteq \mathbb{R}^n$ is a subspace, its dual cone is its orthogonal complement
 $V^* = V^\perp = \{y \mid y^T v = 0 \ \forall \ v \in V\}$
 a subspace is a cone
 • Duality of cones is consistent with duality in subspaces

② FROM DUAL OF NORM CONE TO DUAL NORM

Let $\|\cdot\|$ be a norm on \mathbb{R}^n
 The dual of $K = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$
 is $K^* = \{(u, v) \in \mathbb{R}^{n+1} \mid \|u\|_* \leq v\}$

Where $\|u\|_* = \sup \{u^T x \mid \|x\| \leq 1\}$

Proof: We need to show that $x^T u + tv \geq 0$ whenever $\|x\| \leq t \iff \|u\|_* \leq v$.
 Aside: If this were p-norm for $p \in \mathbb{Z}^+$ then $* = q$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ (2.20)

Let us start by showing that the righthand condition on (u, v) implies the lefthand condition. Suppose $\|u\|_* \leq v$, and $\|x\| \leq t$ for some $t > 0$. (If $t = 0$, x must be zero, so obviously $u^T x + vt \geq 0$.) Applying the definition of the dual norm, and the fact that $\| -x/t \| \leq 1$, we have

$$u^T (-x/t) \leq \|u\|_* \leq v,$$

and therefore $u^T x + vt \geq 0$.

Next we show that the lefthand condition in (2.20) implies the righthand condition in (2.20). Suppose $\|u\|_* > v$, i.e., that the righthand condition does not hold. Then by the definition of the dual norm, there exists an x with $\|x\| \leq 1$ and $x^T u > v$. Taking $t = 1$, we have

$$u^T (-x) + v < 0,$$

which contradicts the lefthand condition in (2.20).