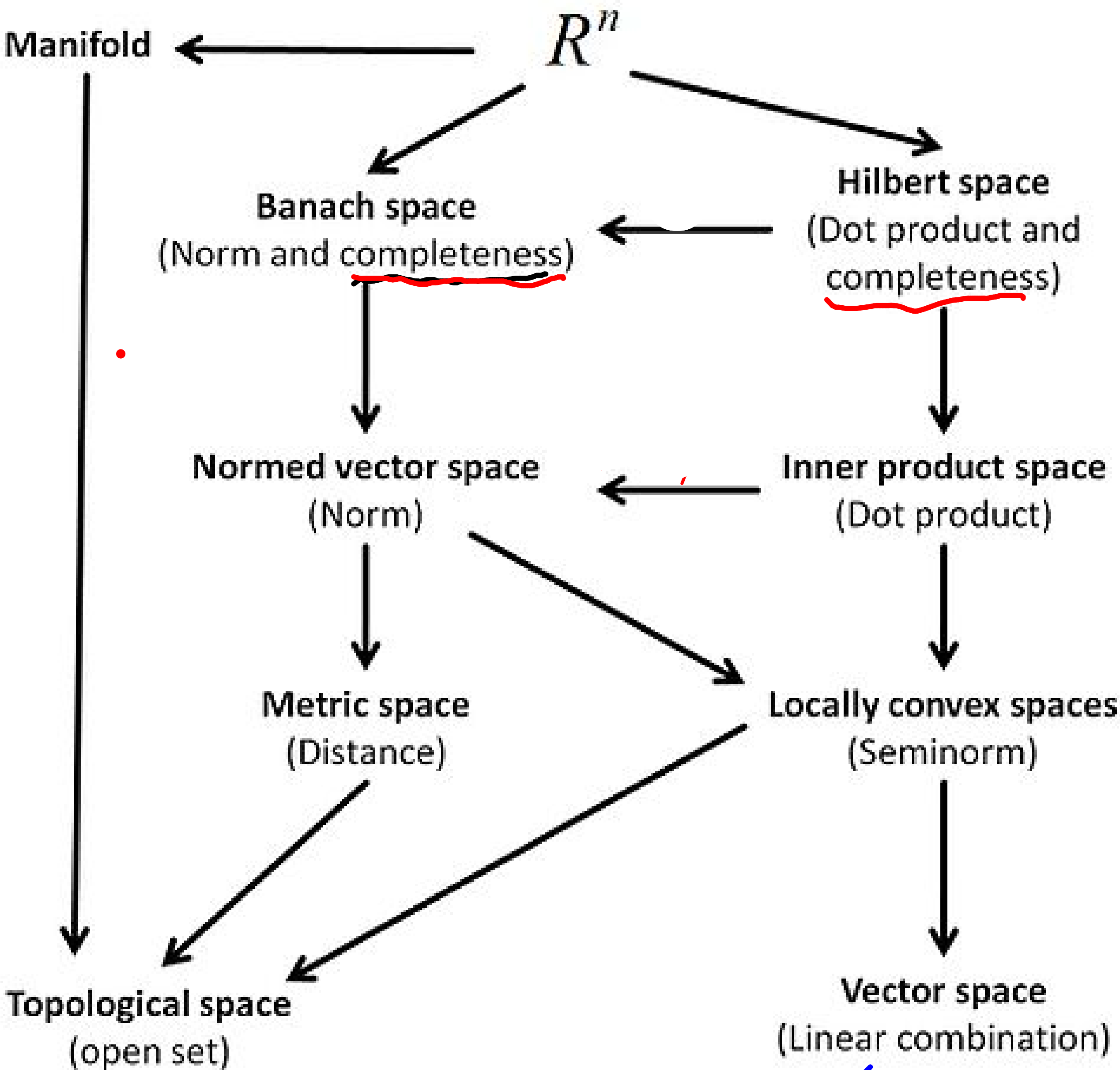
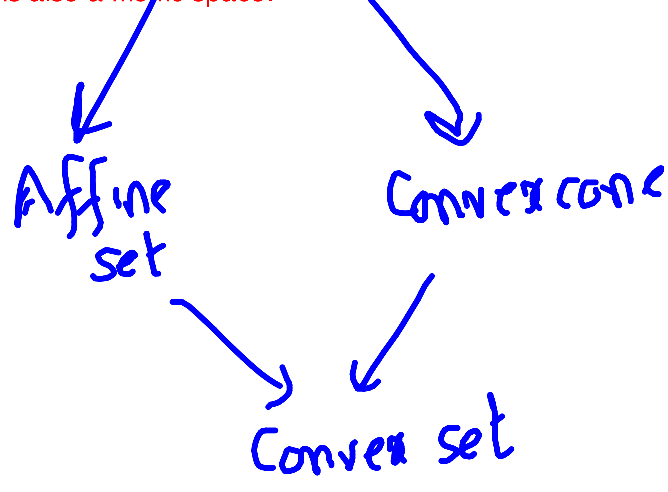


Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



Complete metric space

A metric space S in which every cauchy sequence in S is convergent in S

Claim:

$$C^* = \{ (u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \leq v \}$$

where $\|u\|_*$ = dual norm = $\sup\{ u^T x \mid \|x\|_p \leq 1 \}$

operator norm

we show that

$$\langle x, u \rangle + tv \geq 0 \text{ for } \|x\|_p \leq t \quad \text{A}$$

$$\Updownarrow$$

$$\|u\|_* \leq v \quad \text{B}$$

B $\Rightarrow$ A: Suppose $\|u\|_* \leq v$ & $\|x\| \leq t$ for some $t > 0$ (what happens if $t = 0$)

$\Rightarrow \langle u, -x/t \rangle \leq \|u\|_* \leq v$ ... $\Rightarrow$ A

A $\Rightarrow$ B: Let $\|u\|_* > v$ (ie by contradiction)

$\Rightarrow \exists$ an x with $\|x\| \leq 1$ & $\langle x, u \rangle > v$

Taking $t = 1$, $\langle u, -x \rangle + v < 0$ which contradicts A

Further: If $p \in [1, \infty)$ then $\|u\|_* = \|u\|_q$ s.t $\frac{1}{p} + \frac{1}{q} = 1$ H/W

In particular, Euclidean norm is self dual: $\|u\|_* = \|u\|_2$

EXTRA READING

BEGINS

(e) If X is normed v.s & Y is Banach then $T: X \to Y$ is Banach w.r.t the operator norm

(f) $T: X \to \mathbb{R}$ is called a linear functional

Then dual of X is set of all its linear functionals

Algebraic dual $= \{T \mid T: X \to \mathbb{R}\} = X^{\#}$

If X is finite dimensional, its dual $X^{\#}$ is linearly isomorphic to X

ie if $\{e_1 \ldots e_n\}$ is basis for X

then $\{g_i : X \to \mathbb{R}\}$ s.t

$$g_i\left(\sum_{j=1}^{n} x_j e_j\right) = x_i$$

form the basis for $X^{\#}$ so that

for any $g \in X^{\#}$, $g\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} g(e_i)\, x_i$

$\{g_1, g_2 \ldots g_n\}$ is called the dual basis

L① Examples of T

ⓐ $A : \mathbb{C}^m \to \mathbb{C}^n$ defined by

$Ax = b$ for $x \in \mathbb{C}^m$ & $b \in \mathbb{C}^n$

& $A \in \mathbb{C}^{n \times m}$

- If $A^* = A^{-1}$ then A is called orthogonal
- $\mathbb{C}^m$ is isomorphic to $\mathbb{C}^n$ if $m = n$

$\|A\|$ was discussed in the case of matrices and its form depends on the norm employed in $\mathbb{C}^m$

ⓑ $I : X \to X$ is the identity operator and is bounded for any normed space X

operator norm? Assume p-norm

ⓒ Let $D : C^\infty([0,1]) \to C^\infty([0,1])$ be the differentiation linear operator on normed space of functions with continuous derivatives of all orders

$C^\infty([0,1])$ is normed but NOT banach

$Du = u' \ \forall \ u \in C^\infty([0,1])$

(d) $T: X \to X$ where $X = \{f : C^\infty \to C^\infty\}$

and $T = \frac{d^2}{dx^2} - \frac{d}{x}$

$f_k(x) = e^{kx}$ is an eigenfunction and $k^2 - k$ the corresponding eigenvalue

(e) $T: X \to X$ where $X = \{f : \mathbb{R} \to \mathbb{R}\}$

and $T = \frac{d}{dx}$

$f_\lambda(x) = e^{\lambda x}$ is an eigenfunction and λ the corresponding eigenvalue

(f) $T: C([0,1]) \to C([0,1])$, C being space of cts fns in $[0,1]$ (with $\| \ \|_\infty$)

T is called the Volterra integral operator

If

$$Tf(x) = \int_0^x f(y)\,dy$$

$\|T\|$ with $\| \|_\infty$ on $C([0,1])$

Also, T is bounded

⑨ $T_R : \ell^\infty(S) \to \ell^\infty(S)$ & $T_L : \ell^\infty(S) \to \ell^\infty(S)$

st $T_R(x_1, x_2, x_3, \dots) = \underbrace{(0, x_1, x_2, x_3 \dots)}_{\text{Right shift operator}}$

OR

$T_L(x_1, x_2, x_3, \dots) = \underbrace{(x_2, x_3, \dots)}_{\text{Left shift operator}}$

$\|T_R\| = \|T_L\| = 1$

$\ker(T_R) = \{0\}$ $\quad$ range$(T_R) = \{(0, x_1, \dots) \in \ell^\infty(S)\}$

$\ker(T_L) = \{(x_1, 0, \dots)\}$ $\quad$ range$(T_L) = \ell^\infty(S)$

(h) $T_n : L^2(D) \to \mathbb{C}$ s.t (D is $[a,b]$ for eg)

$$T_n(f) = \frac{1}{\sqrt{2\pi}} \int_{x \in D} f(x) e^{-inx} \, dx$$

maps function f to its n^{th} Fourier coefficient

Then T_n is bounded $(\|T_n\| = 1)$

Q: span $(T_1(f), T_2(f), \ldots, T_n(f) \ldots)$

(g) $T_y : X \to \mathbb{C}$ s.t X & Y are subsets of a Hilbert space and $y \in Y$

eg: $L^2(D)$

n

$T_y(x) = \langle y, x \rangle$ is bounded and

Assume X & Y are subsets of same inner prod space

$$\underbrace{\|T_y\| = \|y\|}_{\text{By Cauchy Schwarz.}}$$

Eg: inner prod space $= (\mathbb{R}, *)$

$X = [a,b]$ $Y = [c,d]$

Specialities of finite dimensional spaces

(i) Every finite dimensional normed vector space is a Banach space

(ii) Every linear operator on a finite dimensional vector space is bounded/continuous

. - . - H/W: Complete

(9) Topological dual $= \{T \mid T: X \to \mathbb{R}\} = X^*$

In finite dimensional case: $X^* = X^\#$ & X^* is isomorphic to X

T is a continuous linear functional

You get specific duals for subsets of vector spaces (such as convex sets, cones and affine sets) by putting restrictions on T.

Eg: If $C \subseteq X$ s.t X is a vector space

$$\boxed{\text{(a) } C^\# = \text{algebraic dual cone} = \{T \in X^\# \mid T(x) \geq 0 \ \forall x \in C\}}$$

in book represented as $\langle T, x \rangle$

(b) Further if X is a topological vector space & $C \subseteq X$

then

$C^* =$ topological dual cone

$= \{ T \in X^* \mid T(x) \geq 0 \ \ \forall x \in C \}$

Claims:

① C^* is always a convex cone (irrespective of whether C is convex or cone or neither)

if $T_1 \in C^*$ & $T_2 \in C^*$ & $\theta_1, \theta_2 \geq 0$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \in X^*$$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \geq 0$$

$\Rightarrow C^*$ is a convex cone

(Similarly $C^\#$ is also always a convex cone)

(2) If X is finite dimensional,

$$C^{\#} = C^{*}$$

since $X^{\#} = X^{*}$

(3) If X is a Hilbert space,

C^{*} is closed .... More properties follow when X is a Hilbert space

(h) Riesz representation theorem:

If $T: X \to \mathbb{R}$ and X is Hilbert and T is bounded, then $\exists$ a unique vector $y \in X$ s.t

$$T(x) = \langle y, x \rangle \quad \forall\, x \in X$$

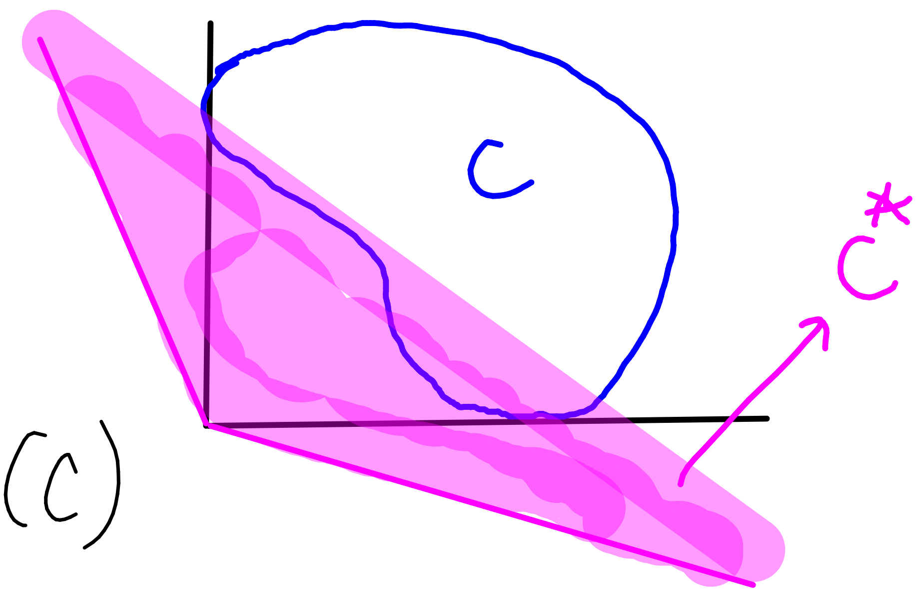
In fact $X^* = \{ T_y(x) = \langle y, x \rangle \mid x \in X \}$ is the dual of X^*

Defines a linear functional in terms of an inner product

Further, X & X^* are isomorphic.

(i) Thus, if X is a Hilbert space over $\mathbb{R}$ as scalars and inner product $\langle , \rangle$, dual cone C^* of a set $C \subseteq X$ is

$$C^* = \{ y \in X : \langle y, x \rangle \geq 0 \ \forall x \in C \}$$



$C^{**} = \text{conichull}(C)$

RECAP THE SPECIAL CASES:

① Vector space $\rightarrow \{x \mid x = \sum_i \alpha_i e_i, (e_1 \dots e_n) : \text{basis}\}$

$\rightarrow \{x \mid \langle a_i, x\rangle = 0 \ \forall i\}$

② Affine space $\rightarrow \{x \mid x = \sum_i \alpha_i e_i \ .. \ \sum \alpha_i = 1\}$

$\rightarrow \{x \mid \langle a_i, x\rangle = b_i \ \forall i\}$

③ closed polytopes $\rightarrow \{x \mid x = \sum_i \alpha_i v_i \quad \sum \alpha_i = 1, \ \alpha_i \in [0,1]\}$

$\rightarrow \{x \mid \langle a_i, x\rangle \geq b_i \ \forall i\}$

The parts in pink deal with characterization of the set in terms of linear operators $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

with $\langle a_i, x\rangle$ viewed as $A(x)$

④ closed polyhedral cone $\rightarrow \{x \mid x = \sum_i \alpha_i v_i \quad \alpha_i \geq 0\}$

$\rightarrow \{x \mid \langle a_i, x\rangle \geq 0 \ \forall i\}$

⑤ closed convex sets $\rightarrow$ Normal defn

$\rightarrow$ intersection of all half spaces containing the set

EXTRA READING

ENDS

Properties of dual cones

① If X is a Hilbert space & $C \subseteq X$ then C^* is a closed convex cone

↳ We have already proved that C^* is a convex cone

↳ C^* = intersection of infinite closed topological half spaces

$$C^* = \bigcap_{x \in C} \{ y \mid y \in X, \langle y, x \rangle \geq 0 \}$$

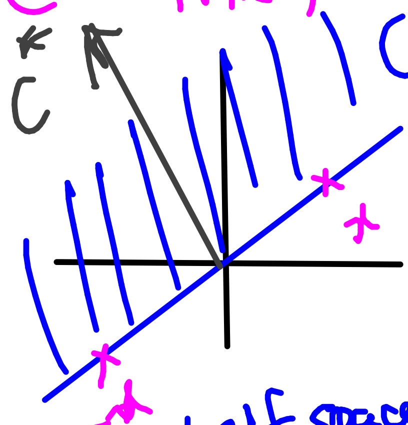
$\Rightarrow C^*$ is closed

② $C_1 \subseteq C_2 \Rightarrow C_2^* \subseteq C_1^*$

③ $\text{interior}(C^*) = \{ y \in X \mid \langle y, x \rangle > 0 \ \forall x \in X \}$

④ If C is a cone and has $\text{int}(C) \neq \emptyset$ the C^* is pointed

↳ ie if $x \in C^*$ & $-x \in C^*$ then $x = 0$



⑤ If C is a cone then closure$(C) = C^{**}$

if C = open half space, C^{**} = closed half space

⑥ If closure of C is pointed then interior$(C^*) \neq \emptyset$

S is called conically spanning set of cone K iff conic$(S) = K$

Positive semidefinite cone

notation:

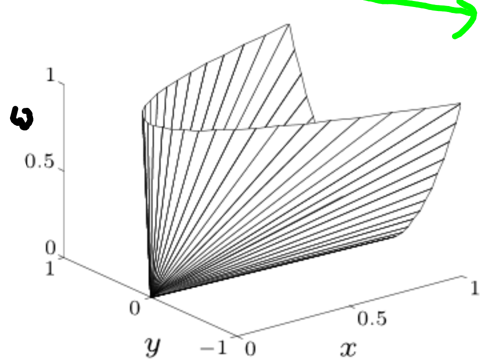
- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices → Easy to prove it is a cone

$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

$\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices → Q: Is it convex? ✓ Is it a cone? ✗ Since $0 \notin S^n_{++}$

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$



Notes abt p.d cone: (or psd cone)

$$S_+^M = \{A \in S^M \mid A \succeq 0\} = \{A \in S^M \mid y^T A y \geq 0 \ \forall \|y\| = 1\}$$

$$= \bigcap_{\|y\|=1} \{A \in S^M \mid \underbrace{\langle yy^T, A \rangle \geq 0}\}$$

$$y^T A y = \sum_j \sum_i y_i a_{ij} y_j = \sum_i \sum_j (y_i y_j) a_{ij}$$

$$= \langle y^T y, A \rangle$$

= intersection of infinite # of half spaces belonging to $R^{M(M+1)/2}$

Cone boundary consists of all singular p.s.d matrices having atleast one 0 eigenvalue

ORIGIN = 0 = matrix with all 0 eigenvalues

Interior consists of all full rank matrices A (rank $A = m$) ie $A \succ 0$

Notes abt p.d cone: (or psd cone)

$$S_+^n = \{A \in S^n \mid A \succeq 0\} = \{A \in S^n \mid y^T A y \geq 0 \ \forall \|y\| = 1\}$$

$$= \bigcap_{\|y\|=1} \{A \in S \mid \langle y^T y, A\rangle \geq 0\}$$

$$y^T A y = \sum_j \sum_i y_i a_{ij} y_j = \sum_i \sum_j (y_i y_j) a_{ij}$$

$$= \langle yy^T, A\rangle = \mathrm{tr}\left((yy^T)^T A\right) = \mathrm{tr}(yy^T A)$$

= Intersection of infinite # of half spaces belonging to $R^{n(n+1)/2}$ [Dual representation]

$y = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \Rightarrow yy^T = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$

H/W: Plot a finite # of halfspaces parametrized by θ

Cone boundary consists of all singular p.s.d matrices having atleast one 0 eigenvalue

H/W

Interior consists of all full rank matrices A (rank $A = m$) ie $A \succ 0$

ORIGIN = 0 = matrix with all 0 eigenvalues

Claim: $(S_+^n)^* = S_+^n$

ie $\langle X, Y\rangle = \mathrm{tr}(X^T Y) = \mathrm{tr}(XY) \geq 0 \ \forall X \in S_+^n$

$\Updownarrow$ iff

$Y \in S_+^n$

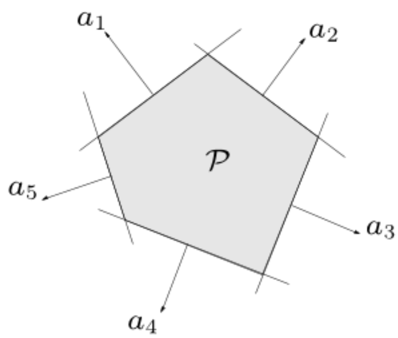
Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)

The Hahn Banach Thm: Any closed convex set in $\mathbb{R}^n$ is equivalent to intersection of all halfspaces that contain it



polyhedron is intersection of finite number of halfspaces and hyperplanes

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}^n_+$ is a convex cone
- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$