Claim:

$$C^* = \{(u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \leq v\}$$

where $\|u\|_*$ = dual norm = $\sup\{u^T x \mid \|x\|_p \leq 1\}$

operator norm

we show that

$$\langle x, u\rangle + tv \geq 0 \text{ for } \|x\|_p \leq t \quad \text{A}$$

$$\Updownarrow$$

$$\|u\|_* \leq v \quad \text{B}$$

B $\Rightarrow$ A: Suppose $\|u\|_* \leq v$ & $\|x\| \leq t$ for some $t > 0$ (what happens if $t = 0$)

$\Rightarrow \langle u, -x/t\rangle \leq \|u\|_* \leq v \ldots \Rightarrow$ A

A $\Rightarrow$ B: Let $\|u\|_* > v$ (ie by contradiction)

$\Rightarrow \exists$ an x with $\|x\| \leq 1$ & $\langle x, u\rangle > v$

Taking $t=1$, $\langle u, -x\rangle + v < 0$ which contradicts A

Further: If $p \in [1,\infty)$ then $\|u\|_* = \|u\|_q$ s.t $\frac{1}{p} + \frac{1}{q} = 1$ H/W

In particular, Eucledian norm is self dual: $\|u\|_* = \|u\|_2$

$$\|u\|_* = \sup_x \{ \langle u, x \rangle \mid \|x\|_p \leq 1 \}$$

① If $u=0$ then both sides are trivially 0. Assume $u \neq 0$

② Holder's inequality has that (proved using Jensen's inequality)

$$\sum_{i=1}^{n} |u_i x_i| \leq \|u\|_q \|x\|_p \quad \text{where } \frac{1}{p}+\frac{1}{q}=1 \ \& \ p,q \in [1,\infty)$$

③ If $\|x\|_p \leq 1$, Using Holder's inequality....

$$\langle u, x \rangle = \sum_{i=1}^{n} u_i x_i \leq \sum_{i=1}^{n} |u_i x_i| \leq \|u\|_q \|x\|_p \leq \|u\|_q$$

Thus: $\|u\|_* = \sup_x \{ \langle u, x \rangle \mid \|x\|_p \leq 1 \} \leq \|u\|_q \rightarrow$ Ⓐ

④ We will show that $\exists\, x$ for which equality holds in Ⓐ

Let $y = \text{sign}(u)|u|^{q-1}$ i.e. $y_i = \text{sign}(u_i)|u_i|^{q-1} \ \forall\, i=1..n$

Then $\langle u, y \rangle = \sum_{i=1}^{n} u_i \,\text{sign}(u_i)|u_i|^{q-1} = \sum_{i=1}^{n} |u_i|^q = \|u\|_q^q \rightarrow$ Ⓑ

⑤ $\|y\|_p^p = \sum_{i=1}^{n} |y_i|^p = \sum_{i=1}^{n} |\text{sign}(u_i)|\,|u_i|^{p(q-1)} = \sum_{i=1}^{n} |u_i|^q = \|u\|_q^q \rightarrow$ Ⓒ

($\because \frac{1}{p}+\frac{1}{q}=1 \Rightarrow p+q=pq \Rightarrow p(q-1)=q$)

⑥ Let $x = \dfrac{y}{\|y\|_p} \Rightarrow \|x\|_p = 1$ $\Big\}$ Since $u \neq 0$, $\|u\|_q^q = \|y\|_p^p \neq 0 \Rightarrow y \neq 0$

Then

$$\sum_{i=1}^{n} u_i x_i = \sum_{i=1}^{n} u_i \frac{y_i}{\|y\|_p} = \frac{1}{\|y\|_p} \sum_{i=1}^{n} u_i y_i$$

$$= \frac{\|u\|_q^q}{\|y\|_p} \quad \text{from (B)}$$

$$= \frac{\|u\|_q^q}{\|u\|_q^{q/p}} \quad \text{from (C)}$$

$$= \underbrace{\|u\|_q^{(pq-q)/p}}_{\frac{1}{p}+\frac{1}{q}=1} = \|u\|_q$$

One could also prove using the method of Lagrange multipliers (https://whoif.files.wordpress.com/2015/05/proof-of-dual-norm-of-lp-norm3.pdf) but that will be jumping ahead of the course!

We have already seen that the class of functions f for fixed domain D & range vector space V themselves form a vector space

$$f : D \to V$$

Next we see another class of functions (convex functions) which is convex cone

3. Convex functions

- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions
- convexity with respect to generalized inequalities

so that $\theta x + (1-\theta) y \in \operatorname{dom} f$ $\forall\, x, y \in \operatorname{dom} f$

Definition

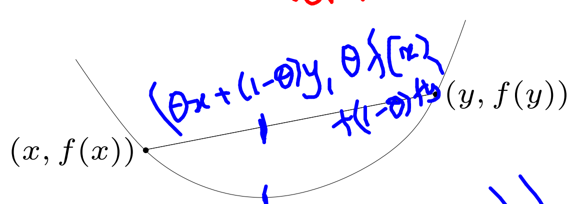
$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

fn at convex combination

$$\underbrace{f(\theta x + (1-\theta)y)}_{\text{fn at convex combination}} \leq \underbrace{\theta f(x) + (1-\theta) f(y)}_{\text{convex combination of fns at } x \,\&\, y}$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$

convex combination of fns at x & y



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 < \theta < 1$

Epigraph and sublevel set

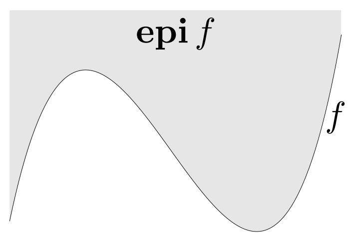
α-sublevel set of $f : \mathbf{R}^n \to \mathbf{R}$:

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \le \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

epigraph of $f : \mathbf{R}^n \to \mathbf{R}$:

$$\mathbf{epi}\, f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \le t\}$$



f is convex if and only if $\mathbf{epi}\, f$ is a convex set

The family of convex functions is a convex cone } Q1

If f is convex, is C_α convex?
If f is convex, is epi$f$ convex? } Q2

Does convexity of C_α or epi$f$ imply convexity of f? } Q3

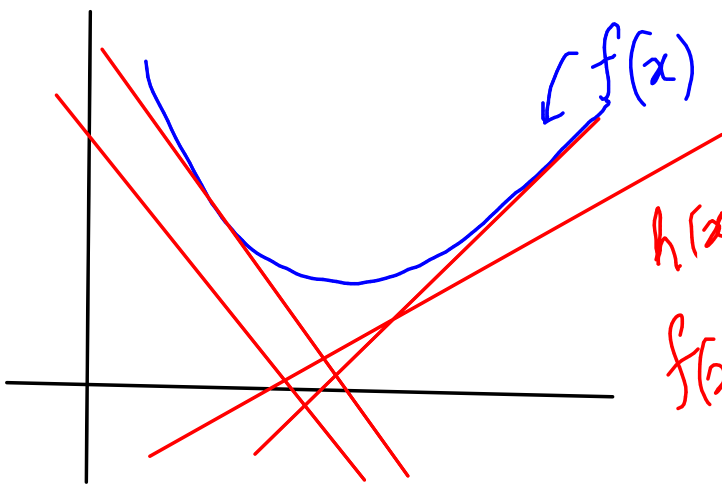
Epi(f) is a convex set iff f is a convex fn

Dual characterization?

$$\boxed{\text{"A closed convex set is the intersection of all half spaces containing it"}}$$

Q: How to characterize the half spaces that contain the epigraph of f?

Soln: Consider all affine functions that lie below epi(f)



$h(x) = \langle a, x \rangle + b$

$f(x) \geq h(x) \;\; \forall x$

The set of all affine h "supporting" this inequality is called the support of f (supp(f))

if epi f is closed ie if f is finite (proper) & lower semicts

Claim: $f(x) = \sup_{h \in \text{supp} f} h(x)$

Epi (f) is a convex set iff f is a convex fn

Dual characterization?

> "A closed convex set is the intersection of all half spaces containing it"

Q: How to characterize the half spaces that contain the epigraph of f?

Soln: Consider the conjugate fn

$$f^*(y) = \sup_{x \in dmn f} \left(\langle y, x \rangle - f(x) \right)$$

$$\inf_{x \in dmn f} f(x) = f^*(0)$$

$$f^*(y) \geq \langle y, x \rangle - f(x) \quad \forall x \in dmn f$$

$$\Rightarrow f^*(y) + f(x) \geq \langle y, x \rangle \quad \forall x \in dmn f$$

Young Fenchel inequality

Epigraph and sublevel set

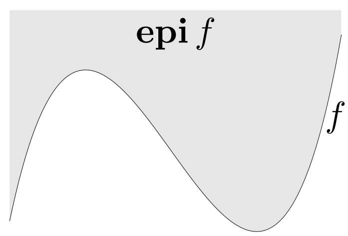
α-**sublevel set** of $f : \mathbf{R}^n \to \mathbf{R}$:

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \le \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

epigraph of $f : \mathbf{R}^n \to \mathbf{R}$:

$$\mathbf{epi}\, f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \le t\}$$



f is convex if and only if $\mathbf{epi}\, f$ is a convex set

More generally: $\bar{f}$ is K-convex iff epi $\bar{f}$ (wrt $\hat{K}$) is a convex set } Q1

Think: When is epi(f) closed? } Q2

When is epi($\bar{f}$) closed?