Epi(f) is a convex set iff f is a convex fn

Dual characterization?

> "A closed convex set is the intersection of all half spaces containing it"

Q: How to characterize the half spaces that contain the epigraph of f?

Soln: Consider the conjugate fn

$$f^*(y) = \sup_{x \in \mathrm{dmn} f} \left(\langle y, x \rangle - f(x) \right)$$

$\inf_{x \in \mathrm{dmn} f} f(x) = f^*(0)$

$$f^*(y) \geq \langle y, x \rangle - f(x) \quad \forall x \in \mathrm{dmn} f$$

$$\Rightarrow f^*(y) + f(x) \geq \langle y, x \rangle \quad \forall x \in \mathrm{dmn} f$$

Young Fenchel inequality

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Dual characterization?

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Soln: Consider the conjugate fn

$$f^*(y) = \sup_{x \in \mathrm{dmn} f} \left(\langle y, x \rangle - f(x) \right)$$

It can be proved that if f is differentiable at x & $\nabla f(x) = g_x$ is gradient at x then

$$f^*(g_x) + f(x) = \langle g_x, x \rangle$$

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Dual characterization?

> "A closed convex set is the intersection of all half spaces containing it"

Q: How to characterize the half spaces that contain the epigraph of f?

Soln: Consider the conjugate fn

$$f^*(y) = \sup_{x \in \mathrm{dmn} f} \left(\langle y, x \rangle - f(x) \right)$$

$$(f^*)^*(x) = \sup_{y \in \mathbb{R}^n} \left\{ \langle y, x \rangle - f^*(y) \right\}$$

We know: $f(x) \geq \langle y, x \rangle - f^*(y) \quad \forall y \in \mathbb{R}^n$

$$\Rightarrow f(x) \geq \sup_{y \in \mathbb{R}^n} \left\{ \langle y, x \rangle - f^*(y) \right\} = (f^*)^*(x)$$

If epi(f) is closed [ie f is proper & lower semi cts] then $f(x) = (f^*)^*(x)$

Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that f is obtained from simple convex functions by operations that preserve convexity

   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

Positive weighted sum & composition with affine function

nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

composition with affine function: $f(Ax + b)$ is convex if f is convex

examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

Pointwise maximum

if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ is convex
- sum of r largest components of $x \in \mathbf{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

is convex ($x_{[i]}$ is ith largest component of x)

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

Pointwise supremum

if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

examples

- support function of a set C: $S_C(x) = \sup_{y \in C} y^T x$ is convex
- distance to farthest point in a set C:

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$

The conjugate function

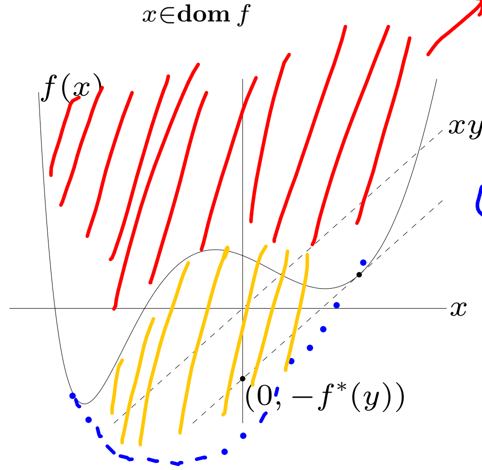
the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$

epigraph of f

epigraph of conjugate of conjugate is convex hull of the epigraph

$\mathrm{epi}((f^*)^*) = \mathrm{conv}(\mathrm{epi}(f))$

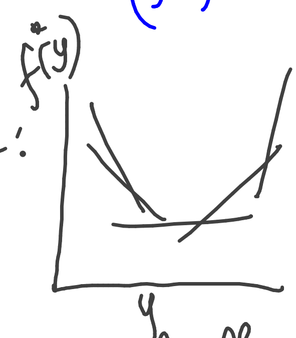


conjugate of conjugate is the "convex envelope" of the function

$(f^*)^* = \mathrm{convenvlp}(f)$

- f^* is convex (even if f is not)
- will be useful in chapter 5

conjugate:



ie

$f(x) \geq \langle x, \tilde{y} \rangle - f^*(\tilde{y}) \quad \forall x \in \mathrm{dom} f$

if $f^{**}(x^*) = \langle x^*, \tilde{y} \rangle - f^*(\tilde{y})$

(supporting hyperplane to convex hull of epi(f) at x^*

ie supporting hyperplane to epigraph of convex envelope of f at x^*)

$f^*(y)$ is supremum of affine fns

if $f^*(\tilde{y}) = \langle x^*, \tilde{y} \rangle - f(x^*)$

Then $\langle x, \tilde{y} \rangle - f^*(\tilde{y}) = h(x)$

is supporting hyperplane to $\mathrm{conv}(\mathrm{epi}(f))$ at x^*

The conjugate function

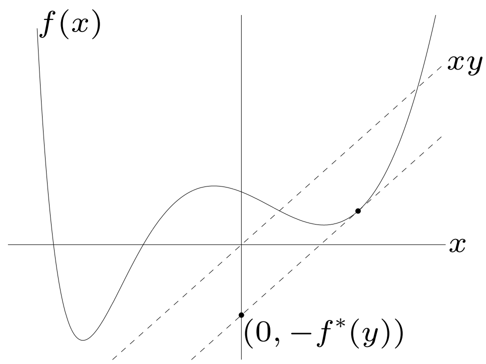
the **conjugate** of a function f is

$$f^*(y) = \sup_{x\in \mathbf{dom}\, f} (y^Tx - f(x))$$

$$\sup_{x\in dom f}\left(g_{y_1}(x) + g_{y_2}(x)\right)$$

$$\alpha\langle x, y_1\rangle - \alpha f(x) + (1-\alpha)\langle x, y_2\rangle - (1-\alpha) f(x)$$

$$\le \sup_{x\in dom f} g_{y_1}(x) + \sup_{x \in dom f} g_{y_2}(x)$$



- f^* is convex (even if f is not) $: f^*(\alpha y_1 + (1-\alpha)y_2) = \sup_{x\in dmn f}\left\{\langle x, \alpha y_1 + (1-\alpha)y_2\rangle - f(x)\right\}$
- will be useful in chapter 5

$$\le \sup_{x\in dmn f} \alpha\left\{\langle x, y_1\rangle - f(x)\right\} + \sup_{x\in dmn f}(1-\alpha)\left\{\langle x, y_2\rangle - f(x)\right\}$$

$$= \alpha f^*(y_1) + (1-\alpha) f^*(y_2)$$

examples

- negative logarithm $f(x) = -\log x$

$$f^*(y) \quad =$$

extended value extension of $\left. f^*(y) = -1-\log(-y) \text{ for } y<0 \right\} = -1-\log(-y)$ if $y<0$ & ∞ if $y \ge 0$

- strictly convex quadratic $f(x) = (1/2)x^TQx$ with $Q \in \mathbf{S}^n_{++}$

$$f^*(y) \quad = \sup_x \left(x^Ty - \tfrac{1}{2}x^TQx\right)$$

$$= \left(y^TQ^{-1}y - \tfrac{1}{2}y^TQ^{-1}y\right)$$

$$= \tfrac{1}{2}y^TQ^{-1}y$$

Necessary condition for min at x, since $x^Ty - \frac{1}{2}x^TQx$ is differentiable

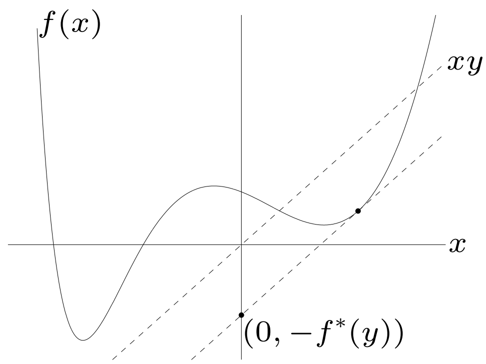
$D_x\left(x^Ty - \frac{1}{2}x^TQx\right) = 0$

$D_x\left(x^Ty - \frac{1}{2}x^TQx\right) = 0$ ie $y - Qx = 0$ ie $y = Qx$ ie $x = Q^{-1}y$

The conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$



- f^* is convex (even if f is not)
- will be useful in chapter 5

examples

- negative logarithm $f(x) = -\log x$

$$\begin{aligned} f^*(y) &= \sup_{x>0}(xy + \log x) \\ &= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- strictly convex quadratic $f(x) = (1/2)x^T Q x$ with $Q \in \mathbf{S}^n_{++}$

$$\begin{aligned} f^*(y) &= \sup_x (y^T x - (1/2)x^T Q x) \\ &= \frac{1}{2} y^T Q^{-1} y \end{aligned}$$

Next question: When is epi(f) closed?

Closed epigraph of convex f

Iff function f is lower-semi-continuous

$f: X \to \mathbb{R}$ is called lower (upper) semi-continuous at $x \in X$ if

$f(x) \le \liminf_{k\to\infty} f(x_k)$ $\left(\ge \limsup_{k\to\infty} f(x_k)\right)$

for every sequence $\{x_k\} \subset X$ that converges to x ① for $X = \mathbb{R}^n$

$\Updownarrow$

Sub-level set $\{x \mid f(x) \le a\}$ is closed for any $a \in \mathbb{R}$ ②

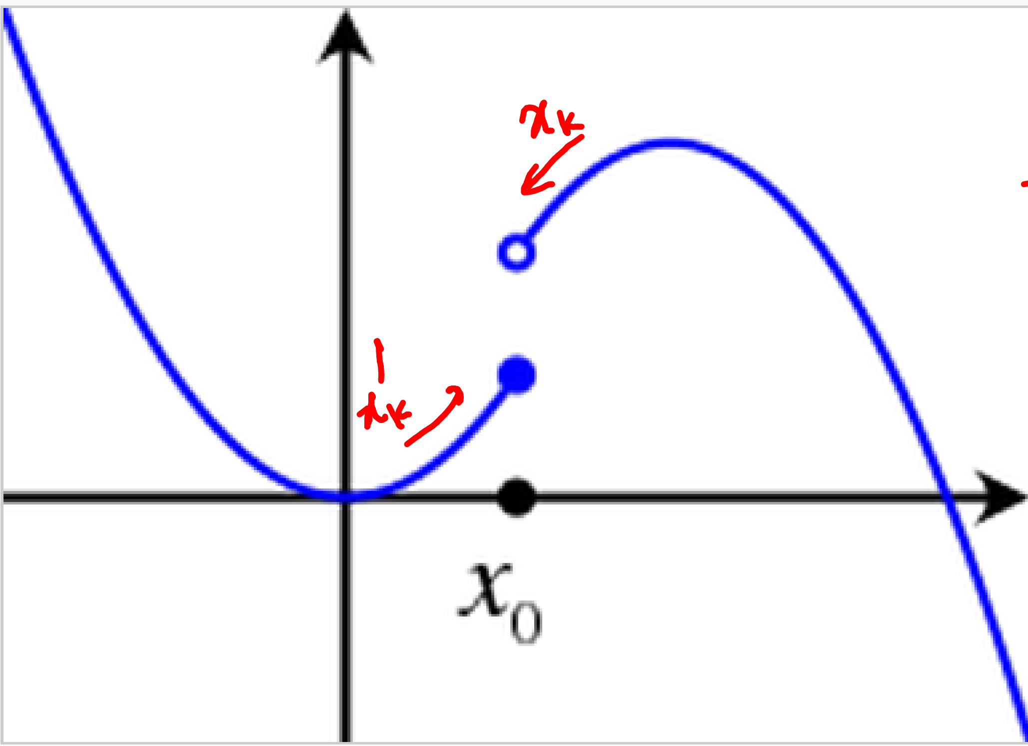
$\Updownarrow$

epigrap(f) is closed ③ (which is generally stated as "f is closed")

- A fn is cts at x_0 iff it is upper & lower semi cts at x_0

Dual characterization in terms of Fenchel conjugate (Legendre transform) of f

- Results in an alternative (to Lagrange) form of duality, called Fenchel duality
- Application: Helps relate Lagrange dual function with primal function



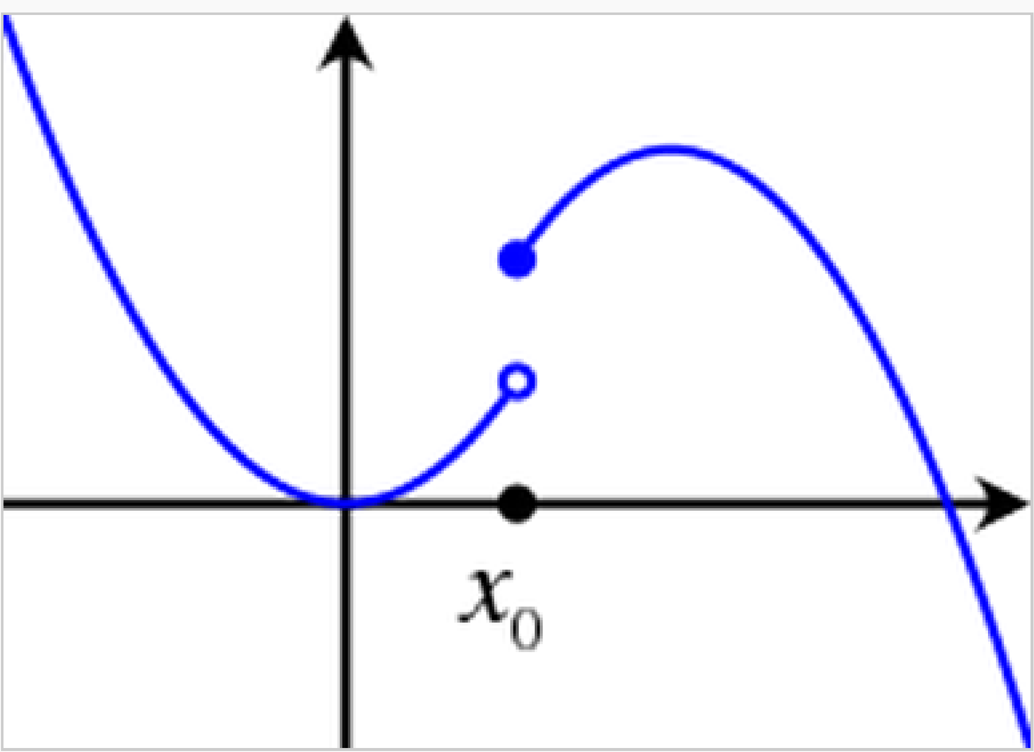
A lower semi-continuous function. The solid blue dot indicates $f(x_0)$.

→ Need not be left cts

→ eg:

$$f(x) = \begin{cases} -1 & \text{if } x<1 \\ -1.5 & \text{if } x=1 \\ -3/4 & \text{if } x>1 \end{cases}$$

$$f(x) = \begin{cases} -\sin\left(\frac{1}{x}\right) & x \neq 0 \\ -1 & x=0 \end{cases}$$



An upper semi-continuous function. The solid blue dot indicates $f(x_0)$.

→ Need not be right cts

Proof: (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2)

(2) $\Rightarrow$ (1): Suppose $\{x \mid f(x) \leq a\}$ is closed $\forall\, a \in \mathbb{R}$ & for proof by contradiction, say, $\exists\, \bar{x}$ s.t. $f(\bar{x}) > \liminf_{k\to\infty} f(x_k)$ & $\{x_k\} \to \bar{x}$

Let $a \in \mathbb{R}$ be s.t.

$$f(\bar{x}) > a > \liminf_{k\to\infty} f(x_k)$$

$\Rightarrow \exists$ subsequence $\{x_k\}_K$ s.t. $f(x_k) \leq a \;\; \forall\, k \in K$

Since $\{x \mid f(x) \leq a\}$ is closed, $\bar{x}$ must belong to $\{x_k\}_K \Rightarrow f(\bar{x}) \leq a$ ... a contradiction!

(1) $\Rightarrow$ (3) If f is lower semi-continuous over $\mathbb{R}^n$ & if $(\bar{x}, \bar{a})$ is limit of $\{(x_k, a_k)\} \subset \text{epi}(f)$ then $f(x_k) \leq a_k$

and taking $\lim_{k\to\infty} \left(f(x_k) \leq a_k \right)$ and using lower semi-continuity of f at $\bar{x}$

$f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k) \leq \bar{a} \Rightarrow (\bar{x}, \bar{a}) \in \text{epi}(f)$, ie $\text{epi}(f)$ is closed!

③ ⇒ ② If $epi(f)$ is closed & $\{x_k\}$ is a sequence that converges to some $\bar{x}$ & belongs to level set $\{x \mid f(x) \le a\}$ for some a, then $(x_k, a) \in epi(f)$ $\forall$ k & $(x_k, a) \to (\bar{x}, a)$. Since $epi(f)$ is closed,

$(\bar{x}, a) \in epi(f) \Rightarrow \bar{x} \in \{x \mid f(x) \le a\}$

$\Rightarrow \{x \mid f(x) \le a\}$ is closed!

---

eg: ① $f(x) = 1$ for $x \in (-\infty, 0)$ is lower (upper) semi-continuous. Is f closed? (ie is $epi(f)$ closed?)

$\tilde{f}(x) = 1$ if $x \in (-\infty, 0)$ & $= \infty$ o/w

Ans: $epi(f) = \{(x,z) \mid f(x) \le z\} = \left\{(x,z) \,\middle|\, \begin{array}{l} x \in (-\infty, 0) \\ z \in [1, \infty) \end{array}\right\}$

is NOT closed! What went wrong??

Recall: f should be lower semi-continuous over $\mathbb{R}^n$ ... In this case f is lower semi-cts only over $(-\infty, 0)$

Soln: Define extended value extension $\tilde{f}$ of f over $\mathbb{R}^n$ ($n=1$ in this example). If $\tilde{f}$ is lower semi cts over $\mathbb{R}^n$, then $epi(\tilde{f})$ is closed!

Unfortunately, $\tilde{f}$ is NOT lower semicts at 0.

Eg: If $\bar{f}(x) = 1$ if $x \in (-\infty, 0)$ & ∞ o/w

Is $\bar{f}(x)$ lower semi-cts on $\mathbb{R}$?

ANS: NO

$epi(\bar{f}) =$

which is

② If $f : X \to (-\infty, \infty)$ & $dom(f)$ is closed & f is lower semicts on $dom(f)$, then $epi(f)$ is closed

Because $\tilde{f}$ will remain lower semi cts eg: If $\tilde{f}(x) = 1$ if $x \in (-\infty, 0)$ & ∞ o/w

Is $\tilde{f}(x)$ lower semi-cts on $\mathbb{R}$?

ANS: No!

$$epi(\tilde{f}) = \{(x,z) \mid x \in (-\infty, 0), z \in [1, \infty)\} \cup \{(x,z) \mid x \in [0,\infty), z = \infty\}$$

which is also NOT closed

② If $f : X \to (-\infty, \infty)$ & $dom(f)$ is closed & f is lower semicts on $dom(f)$, then $epi(f)$ is closed

Extended-value extension

extended-value extension $\tilde{f}$ of f is

$$\tilde{f}(x) = f(x), \quad x \in \mathbf{dom}\, f, \qquad \tilde{f}(x) = \infty, \quad x \notin \mathbf{dom}\, f$$

often simplifies notation; for example, the condition

$$0 \le \theta \le 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \le \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

(as an inequality in $\mathbf{R} \cup \{\infty\}$), means the same as the two conditions

- $\mathbf{dom}\, f$ is convex
- for $x, y \in \mathbf{dom}\, f$,

$$0 \le \theta \le 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

③ $f(x) = 1/x$ if $x > 0$ $\tilde{f}(x)$ is lower semi cts & $epi(\tilde{f})$ is closed!

In summary:

① epi(f) is closed & convex
$\qquad\qquad\;\;\Updownarrow \qquad\qquad \Updownarrow$
$\tilde{f}$ is lower semi-cts & convex

② If f is convex, it is cts on the relative interior of its domain (& ∴ lower semi-cts on the relative interior of its domain)
Discontinuities possible only on relative boundary
H/w (note pt ④)

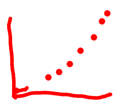
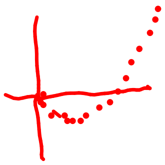
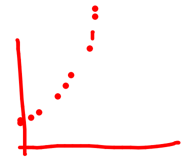
③ Thus, for a convex f, for ensuring closed epi(f), you need to take care of lower semi-continuity of f particularly on the relative boundary of its domain.

④ In particular, if $f: \mathbb{R}^n \to \mathbb{R}$ is convex on $\mathbb{R}^n$ then f (its epigraph) is closed convex & so are its level sets $\{x \mid f(x) \leq a\}\ \forall a$

Examples on R

convex:

- affine: $ax + b$ on **R**, for any $a, b \in \mathbf{R}$ ✓
- exponential: e^{ax}, for any $a \in \mathbf{R}$ AM $\geq$ GM
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$ → for $\alpha < 1$
- powers of absolute value: $|x|^p$ on **R**, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$


concave:

- affine: $ax + b$ on **R**, for any $a, b \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$ $-\log x$ is convex

Most proofs by proving Jensen's inequality!

$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m \times n}$

affine functions are convex and concave; all norms are convex

examples on $\mathbf{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

Triangle ineq & scalar mult

examples on $\mathbf{R}^{m \times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \underbrace{\mathbf{tr}(A^T X)}_{\langle A, X \rangle_F + b} + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

$$\max_v \frac{\|Xv\|_2}{\|v\|_2} = \max_{\|v\|_2 = 1} \|Xv\|_2$$

Proof of Hölder's inequality using Jensen's inequality (ie convexity of $\|x\|_p$ for $p \geq 1$)

Claim: $$\sum_{i=1}^{n} |u_i||x_i| \leq \left(\sum_{i=1}^{n} |u_i|^q\right)^{1/q} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

We will assume $|u_i| > 0$ (why does it suffice to do so?)

Since $\|a\|_p^p$ is convex, by Jensen's inequality:

$$\left(\sum_{i=1}^{n} w_i a_i\right)^p \leq \sum_{i=1}^{n} w_i a_i^p \quad \text{for } a_i, w_i > 0 \ \& \ \sum w_i = 1 \quad (*)$$

Substituting $w_i = \dfrac{|x_i|^p}{\sum_{i=1}^{n} |x_i|^p}$ and $a_i = \dfrac{|u_i||x_i|}{w_i}$ in $(*)$

We get the Hölder's inequality