First-order condition

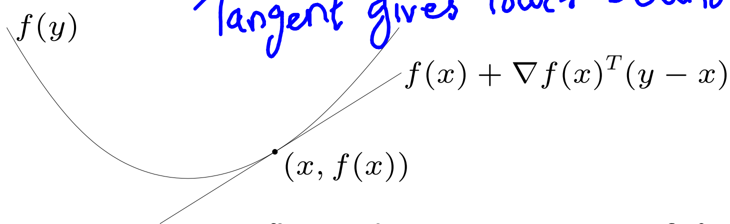
f is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \mathbf{dom}\, f$

1st-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq \underbrace{f(x) + \nabla f(x)^T (y - x)}_{\text{Tangent gives lower bound estimate}} \quad \text{for all } x, y \in \mathbf{dom}\, f$$



first-order approximation of f is global underestimator

Second-order conditions

f is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathbf{dom}\, f$

2nd-order conditions: for twice differentiable f with convex domain

- f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex

Theorem 69 *Let $f : \mathcal{D} \to \Re$ be a convex function on a convex domain $\mathcal{D}$. Any point of locally minimum solution for f is also a point of its globally minimum solution.*

Optimization problem in standard form

$\rightarrow f_0(x)$ is convex

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\
 & h_i(x) = 0, & i = 1, \ldots, p
\end{array}
$$

convex Domain $\mathcal{D}$

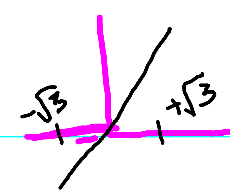
Assume $f_i(x)$ are convex & $h_i(x)$ are affine

$\{x \mid f_i(x) \leq 0\}$ = intersection of sublevel sets of convex fns & hence convex set

$\{x \mid h_i(x) = 0\}$ = intersection of affine sets & hence affine/convex set

$h_i(x) = 0$: Not good enough if h_i is convex since level sets of convex fns not necessarily cvx set

Note: min $f_0(x) = x$ s.t. $x^2 - 3 \leq 0$ has local/global min but min $f_0(x) = x$ s.t. $x \leq 0$ doesnot



Theorem 70 *Let $f : \mathcal{D} \to \Re$ be a strictly convex function on a convex domain $\mathcal{D}$. Then f has a unique point corresponding to its global minimum.* (ie if there exists global minimum)

Proof:

Theorem 69 *Let $f : \mathcal{D} \rightarrow \Re$ be a convex function on a convex domain $\mathcal{D}$. Any point of locally minimum solution for f is also a point of its globally minimum solution.*

If local min x exists, then global min y should exist since o/w if global min does not exist then $\exists\, y$ s.t $f(y) < f(x)$ (since o/w x would have been global min) & then one can prove $\exists\, z = \theta x + (1-\theta) y$ s.t $z \in B_\epsilon$ & $f(z) < f(x)$ .. a contradiction

Proof: Suppose $\mathbf{x} \in \mathcal{D}$ is a point of local minimum and let $\mathbf{y} \in \mathcal{D}$ be a point of global minimum. Thus, $f(\mathbf{y}) < f(\mathbf{x})$. Since $\mathbf{x}$ corresponds to a local minimum, there exists an $\epsilon > 0$ such that

$$\forall\, \mathbf{z} \in \mathcal{D},\ ||\mathbf{z} - \mathbf{x}|| \leq \epsilon \Rightarrow f(\mathbf{z}) \geq f(\mathbf{x})$$

Consider a point $\mathbf{z} = \theta\mathbf{y} + (1-\theta)\mathbf{x}$ with $\theta = \frac{\epsilon}{2||\mathbf{y}-\mathbf{x}||}$. Since $\mathbf{x}$ is a point of local minimum (in a ball of radius ϵ), and since $f(\mathbf{y}) < f(\mathbf{x})$, it must be that $||\mathbf{y} - \mathbf{x}|| > \epsilon$. Thus, $0 < \theta < \frac{1}{2}$ and $\mathbf{z} \in \mathcal{D}$. Furthermore, $||\mathbf{z} - \mathbf{x}|| = \frac{\epsilon}{2}$. Since f is a convex function

$$f(\mathbf{z}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$$

Since $f(\mathbf{y}) < f(\mathbf{x})$, we also have

$$\theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) < f(\mathbf{x})$$

The two equations imply that $f(\mathbf{z}) < f(\mathbf{x})$, which contradicts our assumption that $\mathbf{x}$ corresponds to a point of local minimum. That is f cannot have a point of local minimum, which does not coincide with the point $\mathbf{y}$ of global minimum. $\square$

Theorem 70 *Let $f : \mathcal{D} \rightarrow \Re$ be a strictly convex function on a convex domain $\mathcal{D}$. Then f has a unique point corresponding to its global minimum.* (ie if there exists global minimum)

Proof: Suppose $\mathbf{x} \in \mathcal{D}$ and $\mathbf{y} \in \mathcal{D}$ with $\mathbf{y} \neq \mathbf{x}$ are two points of global minimum. That is $f(\mathbf{x}) = f(\mathbf{y})$ for $\mathbf{y} \neq \mathbf{x}$. The point $\frac{\mathbf{x}+\mathbf{y}}{2}$ also belongs to the convex set $\mathcal{D}$ and since f is strictly convex, we must have

$$f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) < \frac{1}{2} f(\mathbf{x}) + \frac{1}{2} f(\mathbf{y}) = f(\mathbf{x})$$

which is a contradiction. Thus, the point corresponding to the minimum of f must be unique. $\square$

All of this is subject to existence of global min

Eg: $f(x) = -\log x$ is strictly convex without any global min

Definition 41 [Subgradient]: *Let $f : \mathcal{D} \to \Re$ be a convex function defined on a convex set $\mathcal{D}$. A vector $\mathbf{h} \in \Re^n$ is said to be a subgradient of f at the point $\mathbf{x} \in \mathcal{D}$ if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{y} \in \mathcal{D}$. The set of all such vectors is called the subdifferential of f at $\mathbf{x}$.

Theorem 76 *Let $f : \mathcal{D} \to \Re$ be a convex function defined on a convex set $\mathcal{D}$. A point $\mathbf{x} \in \mathcal{D}$ corresponds to a minimum if and only if*

$$\nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \geq 0$$

for all $\mathbf{y} \in \mathcal{D}$.

If $\nabla f(\mathbf{x})$ is nonzero, it defines a supporting hyperplane to $\mathcal{D}$ at the point $\mathbf{x}$. Theorem 77 implies that for a differentiable convex function defined on an open set, every critical point must be a point of (global) minimum.

What do you observe abt $f(x) = x^3 + x$, dmn = $\mathbb{R}$

we note: Whenever

$y > x$ & $f'(x) > 0$

$f'(x)(y-x) \geq 0$

For $f(x) = x^3$, we need to restrict dmn to $D = \{x \mid x \geq \alpha\}$ for $\alpha > 0$

More generally: for any 2 x & y if $\nabla f(x)(y-x) \geq 0$ then $f(y) \geq f(x)$

Claim: if $\nabla f(x)(y-x) \geq 0$ for any 2 x&$y \Rightarrow f(y) \geq f(x)$

& if $\nabla f(x) = 0$ then $\nabla f(x)(y-x) = 0$ $\forall y$

$\Rightarrow f(y) \geq f(x)$ $\forall y$

Definition 41 [Subgradient]: *Let $f : \mathcal{D} \rightarrow \Re$ be a convex function defined on a convex set $\mathcal{D}$. A vector $\mathbf{h} \in \Re^n$ is said to be a subgradient of f at the point $\mathbf{x} \in \mathcal{D}$ if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{y} \in \mathcal{D}$. The set of all such vectors is called the subdifferential of f at $\mathbf{x}$.

Theorem 76 *Let $f : \mathcal{D} \rightarrow \Re$ be a convex function defined on a convex set $\mathcal{D}$. A point $\mathbf{x} \in \mathcal{D}$ corresponds to a minimum if and only if*

$$\boxed{\nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \geq 0}$$

for all $\mathbf{y} \in \mathcal{D}$.

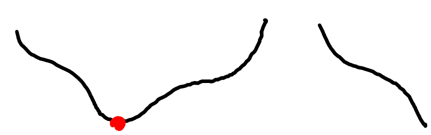
Variational inequality (VI) representation for a convex opt problem

If $\nabla f(\mathbf{x})$ is nonzero, it defines a supporting hyperplane to $\mathcal{D}$ at the point $\mathbf{x}$. Theorem 77 implies that for a differentiable convex function defined on an open set, every critical point must be a point of (global) minimum.

Pseudo convex fn

f is pseudo convex if whenever

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x)$$



① Every convex function is pseudo convex

② Every pseudo convex fn is quasi convex

ie $f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$

$\forall\ \theta \in [0,1]$ & $\forall\ x, y \in \text{dom} f$

③ x^* is pt of local minimum of f iff it is a stationary pt of f ie $\nabla f(x^*) = 0$

Note: $f(x) = x^3 + \alpha x$ is pseudo convex $\forall \alpha > 0$
$D = \mathbb{R}$

$f(x) = x^3$ is quasi convex BUT NOT pseudo convex since $f'(0) = 0$ even though 0 is NOT pt of global min
$D = \mathbb{R}$

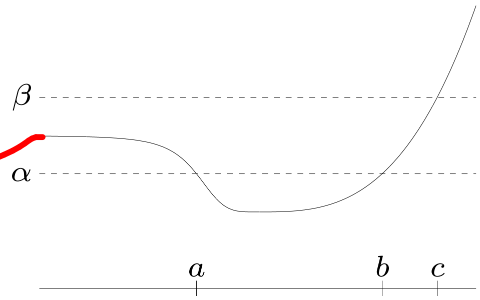
Quasiconvex functions

$f : \mathbf{R}^n \to \mathbf{R}$ is quasiconvex if $\mathbf{dom}\, f$ is convex and the sublevel sets

$$S_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \le \alpha\}$$

are convex for all α

Because fn is increasing here, sublevel sets won't be disjoint unions



- f is quasiconcave if $-f$ is quasiconvex
- f is quasilinear if it is quasiconvex and quasiconcave → All monotone fns are quasilinear

Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbf{R}$
- $\operatorname{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \ge x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbf{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbf{R}^2_{++}$
- linear-fractional function

$$f(x) = \frac{a^T x + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

  is quasilinear
- distance ratio

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \qquad \mathbf{dom}\, f = \{x \mid \|x - a\|_2 \le \|x - b\|_2\}$$

  is quasiconvex

internal rate of return

- cash flow $x = (x_0, \ldots, x_n)$; x_i is payment in period i (to us if $x_i > 0$)
- we assume $x_0 < 0$ and $x_0 + x_1 + \cdots + x_n > 0$
- present value of cash flow x, for interest rate r:

$$\mathrm{PV}(x, r) = \sum_{i=0}^{n} (1+r)^{-i} x_i$$

- internal rate of return is smallest interest rate for which $\mathrm{PV}(x, r) = 0$:

$$\mathrm{IRR}(x) = \inf\{r \geq 0 \mid \mathrm{PV}(x, r) = 0\}$$

IRR is quasiconcave: superlevel set is intersection of halfspaces

$$\mathrm{IRR}(x) \geq R \quad \Longleftrightarrow \quad \sum_{i=0}^{n} (1+r)^{-i} x_i \geq 0 \text{ for } 0 \leq r \leq R$$

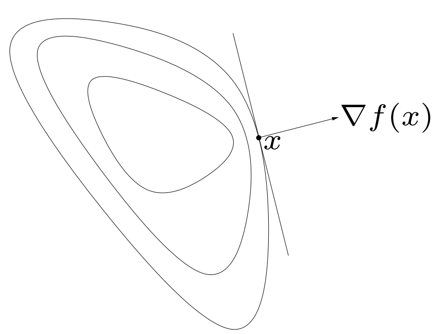
Properties

modified Jensen inequality: for quasiconvex f

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta) y) \leq \max\{f(x), f(y)\}$$

first-order condition: differentiable f with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \quad \Longrightarrow \quad \nabla f(x)^T (y - x) \leq 0$$



sums of quasiconvex functions are not necessarily quasiconvex

Definition 41 [Subgradient]: *Let $f : \mathcal{D} \to \Re$ be a convex function defined on a convex set $\mathcal{D}$. A vector $\mathbf{h} \in \Re^n$ is said to be a subgradient of f at the point $\mathbf{x} \in \mathcal{D}$ if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{y} \in \mathcal{D}$. The set of all such vectors is called the subdifferential of f at $\mathbf{x}$.

Theorem 76 *Let $f : \mathcal{D} \to \Re$ be a convex function defined on a convex set $\mathcal{D}$. A point $\mathbf{x} \in \mathcal{D}$ corresponds to a minimum if and only if*

$$\nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \geq 0$$

for all $\mathbf{y} \in \mathcal{D}$.

If $\nabla f(\mathbf{x})$ is nonzero, it defines a supporting hyperplane to $\mathcal{D}$ at the point $\mathbf{x}$. Theorem 77 implies that for a differentiable convex function defined on an open set, every critical point must be a point of (global) minimum.

Theorem 77 *Let $f : \mathcal{D} \to \Re$ be differentiable and convex on an open convex domain $\mathcal{D} \subseteq \Re^n$. Then $\mathbf{x}$ is a critical point of f if and only if it is a (global) minimum.*

Theorem 78 *Let $f : \mathcal{D} \to \Re$ with $\mathcal{D} \subseteq \Re^n$ be differentiable on the convex set $\mathcal{D}$. Then,*

1. *f is convex on $\mathcal{D}$ if and only if is its gradient ∇f is monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$*

$$\left(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\right)^T (\mathbf{x} - \mathbf{y}) \geq 0 \tag{4.53}$$

2. *f is strictly convex on $\mathcal{D}$ if and only if is its gradient ∇f is strictly monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$ with $\mathbf{x} \neq \mathbf{y}$,*

$$\left(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\right)^T (\mathbf{x} - \mathbf{y}) > 0 \tag{4.54}$$

3. *f is uniformly or strongly convex on $\mathcal{D}$ if and only if is its gradient ∇f is uniformly monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$,*

$$\left(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\right)^T (\mathbf{x} - \mathbf{y}) \geq c||\mathbf{x} - \mathbf{y}||^2 \tag{4.55}$$

for some constant $c > 0$.

Necessity: Suppose f is uniformly convex on $\mathcal{D}$. Then from theorem 75, we know that for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y}-\mathbf{x}) - \frac{1}{2}c||\mathbf{y}+\mathbf{x}||^2$$
$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla^T f(\mathbf{y})(\mathbf{x}-\mathbf{y}) - \frac{1}{2}c||\mathbf{x}+\mathbf{y}||^2$$

Adding the two inequalities, we get (4.55). If f is convex, the inequalities hold with $c = 0$, yielding (4.54). If f is strictly convex, the inequalities will be strict, yielding (4.54).

Sufficiency: Suppose ∇f is monotone. For any fixed $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, consider the function $\phi(t) = f(\mathbf{x} + t(\mathbf{y}-\mathbf{x}))$. By the mean value theorem applied to $\phi(t)$, we should have for some $t \in (0, 1)$,

$$\phi(1) - \phi(0) = \phi'(t) \tag{4.56}$$

Letting $\mathbf{z} = \mathbf{x} + t(\mathbf{y}-\mathbf{x})$, (4.56) translates to

$$f(\mathbf{y}) - f(\mathbf{x}) = \nabla^T f(\mathbf{z})(\mathbf{y}-\mathbf{x}) \tag{4.57}$$

Also, by definition of monotonicity of ∇f, (from (4.53)),

$$\left(\nabla f(\mathbf{z}) - \nabla f(\mathbf{x})\right)^T(\mathbf{y}-\mathbf{x}) = \frac{1}{t}\left(\nabla f(\mathbf{z}) - \nabla f(\mathbf{x})\right)^T(\mathbf{z}-\mathbf{x}) \geq 0 \tag{4.58}$$

Combining (4.57) with (4.58), we get,

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &= \left(\nabla f(\mathbf{z}) - f(\mathbf{x})\right)^T(\mathbf{y}-\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y}-\mathbf{x}) \\ &\geq \nabla^T f(\mathbf{x})(\mathbf{y}-\mathbf{x}) \end{aligned} \tag{4.59}$$

By theorem 75, this inequality proves that f is convex. Strict convexity can be similarly proved by using the strict inequality in (4.58) inherited from strict monotonicity, and letting the strict inequality follow through to (4.59). For the case of strong convexity, from (4.55), we have

$$\begin{aligned} \phi'(t) - \phi'(0) &= \left(\nabla f(\mathbf{z}) - f(\mathbf{x})\right)^T(\mathbf{y}-\mathbf{x}) \\ = \frac{1}{t}\left(\nabla f(\mathbf{z}) - f(\mathbf{x})\right)^T(\mathbf{z}-\mathbf{x}) &\geq \frac{1}{t}c||\mathbf{z}-\mathbf{x}||^2 = ct||\mathbf{y}-\mathbf{x}||^2 \end{aligned} \tag{4.60}$$

$$\phi(1) - \phi(0) - \phi'(0) = \int_0^1 [\phi'(t) - \phi'(0)]dt \geq \frac{1}{2}c||\mathbf{y}-\mathbf{x}||^2 \tag{4.61}$$

which translates to

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y}-\mathbf{x}) + \frac{1}{2}c||\mathbf{y}-\mathbf{x}||^2$$

Theorem 79 *A twice differential function $f : \mathcal{D} \rightarrow \Re$ for a nonempty open convex set $\mathcal{D}$*

1. *is convex if and only if its domain is convex and its Hessian matrix is positive semidefinite at each point in $\mathcal{D}$. That is*

$$\nabla^2 f(\mathbf{x}) \succeq 0 \quad \forall\, \mathbf{x} \in \mathcal{D} \tag{4.62}$$

2. *is strictly convex if its domain is convex and its Hessian matrix is positive definite at each point in $\mathcal{D}$. That is*

$$\nabla^2 f(\mathbf{x}) \succ 0 \quad \forall\, \mathbf{x} \in \mathcal{D} \tag{4.63}$$

3. *is uniformly convex if and only if its domain is convex and its Hessian matrix is uniformly positive definite at each point in $\mathcal{D}$. That is, for any $\mathbf{v} \in \Re^n$ and any $\mathbf{x} \in \mathcal{D}$, there exists a $c > 0$ such that*

$$\mathbf{v}^T \nabla^2 f(\mathbf{x}) \mathbf{v} \geq c||\mathbf{v}||^2 \tag{4.64}$$

In other words

$$\nabla^2 f(\mathbf{x}) \succeq cI_{n \times n}$$

where $I_{n \times n}$ is the $n \times n$ identity matrix and $\succeq$ corresponds to the positive semidefinite inequality. That is, the function f is strongly convex iff $\nabla^2 f(\mathbf{x}) - cI_{n \times n}$ is positive semidefinite, for all $\mathbf{x} \in \mathcal{D}$ and for some constant $c > 0$, which corresponds to the positive minimum curvature of f.

Proof: We will prove only the first statement in the theorem; the other two statements are proved in a similar manner.

Necessity: Suppose f is a convex function, and consider a point $\mathbf{x} \in \mathcal{D}$. We will prove that for any $\mathbf{h} \in \Re^n$, $\mathbf{h}^T \nabla^2 f(\mathbf{x})\mathbf{h} \geq 0$. Since f is convex, by theorem 75, we have

$$f(\mathbf{x} + t\mathbf{h}) \geq f(\mathbf{x}) + t\nabla^T f(\mathbf{x})\mathbf{h} \tag{4.65}$$

Consider the function $\phi(t) = f(\mathbf{x} + t\mathbf{h})$ considered in theorem 71, defined on the domain $\mathcal{D}_\phi = [0, 1]$. Using the chain rule,

$$\phi'(t) = \sum_{i=1}^{n} f_{x_i}(\mathbf{x} + t\mathbf{h})\frac{dx_i}{dt} = \mathbf{h}^T.\nabla f(\mathbf{x} + t\mathbf{h})$$

Since f has partial and mixed partial derivatives, ϕ' is a differentiable function of t on $\mathcal{D}_\phi$ and

$$\phi''(t) = \mathbf{h}^T \nabla^2 f(\mathbf{x} + t\mathbf{h})\mathbf{h}$$

Since ϕ and ϕ' are continous on $\mathcal{D}_\phi$ and ϕ' is differentiable on $int(\mathcal{D}_\phi)$, we can make use of the Taylor's theorem (45) with $n = 3$ to obtain:

$$\phi(t) = \phi(0) + t.\phi'(0) + t^2.\frac{1}{2}\phi''(0) + O(t^3)$$

Writing this equation in terms of f gives

$$f(\mathbf{x} + t\mathbf{h}) = f(\mathbf{x}) + t\mathbf{h}^T \nabla f(\mathbf{x}) + t^2\frac{1}{2}h^T \nabla^2 f(\mathbf{x})\mathbf{h} + O(t^3)$$

In conjunction with (4.65), the above equation implies that

$$\frac{t^2}{2}h^T \nabla^2 f(\mathbf{x})\mathbf{h} + O(t^3) \geq 0$$

Dividing by t^2 and taking limits as $t \to 0$, we get

$$h^T \nabla^2 f(\mathbf{x})\mathbf{h} \geq 0$$

Sufficiency: Suppose that the Hessian matrix is positive semidefinite at each point $\mathbf{x} \in \mathcal{D}$. Consider the same function $\phi(t)$ defined above with $\mathbf{h} = \mathbf{y} - \mathbf{x}$ for $\mathbf{y}, \mathbf{x} \in \mathcal{D}$. Applying Taylor's theorem (45) with $n = 2$ and $a = 0$, we obtain,

$$\phi(1) = \phi(0) + t.\phi'(0) + t^2.\frac{1}{2}\phi''(c)$$

for some $c \in (0, 1)$. Writing this equation in terms of f gives

$$f(\mathbf{x}) = f(\mathbf{y}) + (\mathbf{x} - \mathbf{y})^T \nabla f(\mathbf{y}) + \frac{1}{2}(\mathbf{x} - \mathbf{y})^T \nabla^2 f(\mathbf{z})(\mathbf{x} - \mathbf{y})$$

where $\mathbf{z} = \mathbf{y} + c(\mathbf{x} - \mathbf{y})$. Since $\mathcal{D}$ is convex, $\mathbf{z} \in \mathcal{D}$. Thus, $\nabla^2 f(\mathbf{z}) \succeq 0$. It follows that

Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that f is obtained from simple convex functions by operations that preserve convexity

   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

Positive weighted sum & composition with affine function

nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

composition with affine function: $f(Ax + b)$ is convex if f is convex

examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$