Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

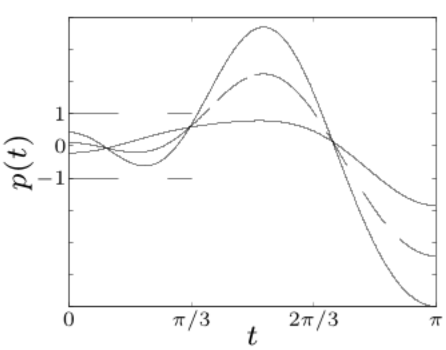
the intersection of (any number of) convex sets is convex

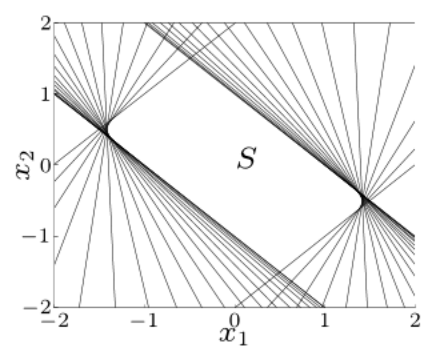
example:

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\} = \bigcap_{|t| \leq \frac{\pi}{3}} \underbrace{\{x \in R^n \mid p_t \leq 1\}}_{\text{half space}}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:





Affine function

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

elementwise inequality

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \le (c^T x)^2,\ c^T x \ge 0\}$ (with $P \in \mathbf{S}^n_+$)

$f(\{x \mid x_1 A_1 + \ldots x_m A_m < B\}) = f(\{x \mid Ax \leq b\})$

Affine fn that serializes the rows of a matrix into a row vector

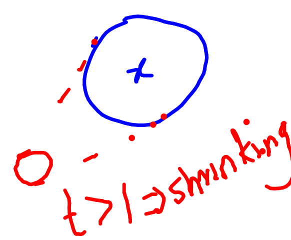
Perspective and linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

midsem Q5 invoked a perspective fn in the proof

$$P(x, t) = x/t, \qquad \mathbf{dom}\, P = \{(x, t) \mid t > 0\}$$

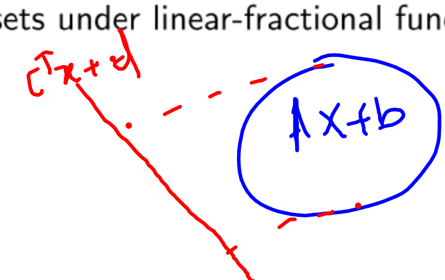
images and inverse images of convex sets under perspective are convex



linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex



example of a linear-fractional function

$$f(x) = \underbrace{\frac{1}{x_1 + x_2 + 1}}_{c^T x + d} x$$