Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that f is obtained from simple convex functions by operations that preserve convexity

   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

For each convexity preserving operation try the set operation on epigraph (H/W)

Positive weighted sum & composition with affine function

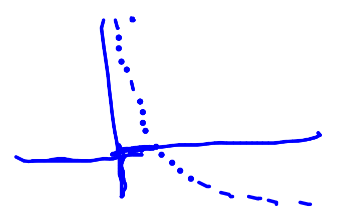
nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

composition with affine function: $f(Ax + b)$ is convex if f is convex

examples

- log barrier for linear inequalities



log is concave

$$f(x) = -\underbrace{\sum_{i=1}^{m} \overbrace{\log(\underbrace{b_i - a_i^T x}_{\text{affine transform}})}^{\text{log is concave}}}_{\text{sum}}, \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

$\min_x f(x)$

s.t. $a_i^T x \leq b_i$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

Pointwise maximum

if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ is convex
- sum of r largest components of $x \in \mathbf{R}^n$:

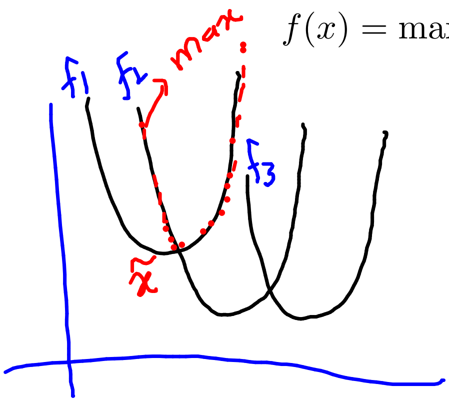
$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]} = \max_{\sigma} \left(\sum_{i=1}^{r} x_{\sigma[i]} \right)$$

σ is a permutation

is convex ($x_{[i]}$ is ith largest component of x)

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}$$



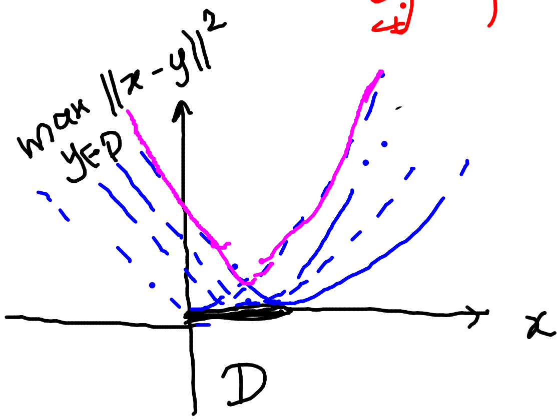
Q: What abt subgradients at $\tilde{x}$ $\partial f(\tilde{x})$?

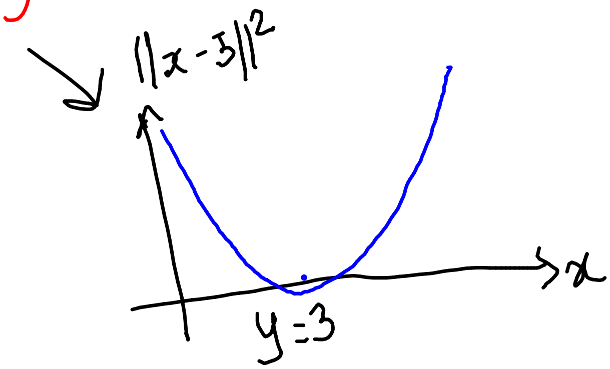
Ans: $\partial f(\tilde{x}) = \text{conv.hull}\left(\nabla f_1(\tilde{x}), \nabla f_2(\tilde{x})\right)$

Q: How abt max over infinite indices?

$\max_{y \in D} f_y(x)$ where $f(x,y)$ is convex in x

E.g. $f(x,y) = \|x-y\|^2$





What if D were union of disjoint intervals?

Pointwise supremum

if $f(x,y)$ is convex in x for each $y \in \mathcal{A}$, then

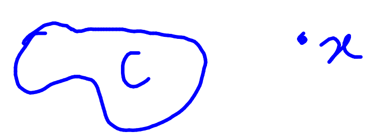
$$g(x) = \sup_{y \in \mathcal{A}} f(x,y)$$

is convex

examples

- support function of a set C: $S_C(x) = \sup_{y \in C} y^T x$ is convex } Recall convex conjugate
- distance to farthest point in a set C:

$$f(x) = \sup_{y \in C} \|x - y\|$$



- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

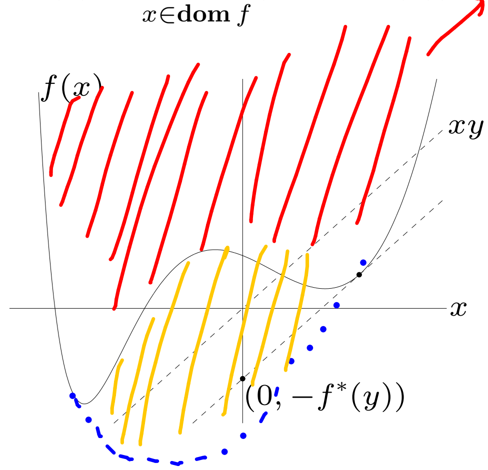
$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$ } Spectral matrix norm

The conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$

epigraph of f



$(f^*)^* = \text{convenvlp}(f)$

- f^* is convex (even if f is not)
- will be useful in chapter 5

conjugate:

ie

$f(x) \geq \langle x, \tilde{y} \rangle - f^*(\tilde{y}) \quad \forall x \in \text{dom} f$

f

$f^{**}(x^*) = \langle x^*, \tilde{y} \rangle - f^*(\tilde{y})$

(supporting hyperplane to convex hull of epi(f) at x^*

ie supporting hyperplane to epigraph of convex envelope of f at x^*)

$f^*(y)$ is supremum of affine fns

if $f^*(\tilde{y}) = \langle x^*, \tilde{y} \rangle - f(x^*)$

Then $\langle x, \tilde{y} \rangle - f^*(\tilde{y}) = h(x)$

is supporting hyperplane to conv(epi(f)) at x^*

The conjugate function

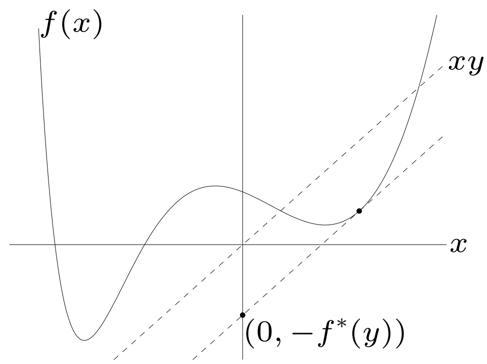
the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$

$$\sup_{x \in \mathrm{dom}\, f} \left(g_{y_1}(x) + g_{y_2}(x) \right)$$

$$\alpha \langle x, y_1 \rangle - \alpha f(x) + (1-\alpha)\langle x, y_2 \rangle - (1-\alpha) f(x)$$

$$\leq \sup_{x \in \mathrm{dom}\, f} g_{y_1}(x) + \sup_{x \in \mathrm{dom}\, f} g_{y_2}(x)$$



- f^* is convex (even if f is not) $: f^*(\alpha y_1 + (1-\alpha) y_2) = \sup_{x \in \mathrm{dom}\, f} \left\{ \langle x, \alpha y_1 + (1-\alpha) y_2 \rangle - f(x) \right\}$
- will be useful in chapter 5

$$\leq \sup_{x \in \mathrm{dom}\, f} \alpha \left\{ \langle x, y_1 \rangle - f(x) \right\} + \sup_{x \in \mathrm{dom}\, f} (1-\alpha) \left\{ \langle x, y_2 \rangle - f(x) \right\}$$

$$= \alpha f^*(y_1) + (1-\alpha) f^*(y_2)$$

examples

- negative logarithm $f(x) = -\log x$

$$f^*(y) \quad =$$

extended value extension of $\left. f^*(y) = -1 - \log(-y) \text{ for } y < 0 \right\} = -1 - \log(-y)$ if $y < 0$ & ∞ if $y \geq 0$

- strictly convex quadratic $f(x) = (1/2) x^T Q x$ with $Q \in \mathbf{S}^n_{++}$

$$f^*(y) \quad = \sup_x \left(x^T y - \frac{1}{2} x^T Q x \right)$$

$$= \left(y^T Q^{-1} y - \frac{1}{2} y^T Q^{-1} y \right)$$

$$= \frac{1}{2} y^T Q^{-1} y$$

Necessary condition for min of $x^T y - \frac{1}{2} x^T Q x$ at x, since $x^T y - \frac{1}{2} x^T Q x$ is differentiable, is

$D_x \left(x^T y - \frac{1}{2} x^T Q x \right) = 0$

ie $y - Qx = 0$ ie $y = Qx$ ie $x = Q^{-1} y$

Composition with scalar functions

composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

$$f(x) = h(g(x))$$

f is convex if h is convex, g is convex & h is increasing

- proof $f''(x) = h''(g(x))(g'(x))^2 + g''(x)\, h'(g(x))$ $(n=1)$

$$\nabla^2 f(x) = h''(g(x))\, \nabla g(x) \nabla^T g(x) + \nabla^2 g(x)\, h'(g(x))$$ (Assume differentiability)

examples

Composition with scalar functions

composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

$$f(x) = h(g(x))$$

f is convex if $\begin{array}{l} g \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing} \end{array}$

- proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- note: monotonicity must hold for extended-value extension $\tilde{h}$

examples

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive

Vector composition

composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

f is convex if $\begin{array}{l} g_i \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing in each argument} \\ g_i \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing in each argument} \end{array}$

proof (for $n = 1$, differentiable g, h)

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)$$

examples

- $\sum_{i=1}^m \log g_i(x)$ is concave if g_i are concave and positive
- $\log \sum_{i=1}^m \exp g_i(x)$ is convex if g_i are convex

Minimization

if $f(x, y)$ is convex in (x, y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex

examples

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$ with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \qquad C \succ 0$$

  minimizing over y gives $g(x) = \inf_y f(x, y) = x^T (A - BC^{-1} B^T) x$

  g is convex, hence Schur complement $A - BC^{-1}B^T \succeq 0$

- distance to a set: $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex

Perspective

the **perspective** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the function $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$,

$$g(x, t) = t f(x/t), \qquad \mathbf{dom}\, g = \{(x, t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$$

g is convex if f is convex

examples

- $f(x) = x^T x$ is convex; hence $g(x, t) = x^T x / t$ is convex for $t > 0$
- negative logarithm $f(x) = -\log x$ is convex; hence relative entropy $g(x, t) = t \log t - t \log x$ is convex on $\mathbf{R}^2_{++}$
- if f is convex, then

$$g(x) = (c^T x + d) f\left((Ax + b)/(c^T x + d)\right)$$

  is convex on $\{x \mid c^T x + d > 0,\ (Ax + b)/(c^T x + d) \in \mathbf{dom}\, f\}$