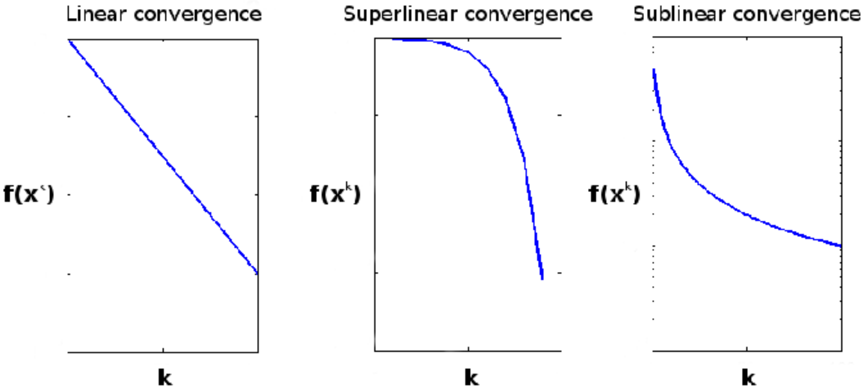
Convergence Analysis

Instructor: Prof. Ganesh Ramakrishnan

Convergence



R-convergence

- Let us consider the convergence result we got by assuming Lipschitz continuity with backtracking and exact line searches:

(& convexity)

$$f(x^k) - f(x^*) \leq \frac{\left\| x^{(0)} - x^* \right\|^2}{2tk}$$

- We will characterize this using **R-convergence**
- 'R' here stands for 'root', as we are looking at convergence rooted at x^*

Q-convergence

- We say that the sequence $s^1, \dots, s^k$ is **R-linearly** convergent if $\left\| s^k - s^* \right\| \leq v^k$, $\forall k$, and $\left\{ v^k \right\}$ converges **Q-linearly** to zero
- $v^1, \dots, v^k$ is Q-linearly convergent if

$$\frac{\left\| v^{k+1} - v^* \right\|}{\left\| v^k - v^* \right\|} \leq r \in (0,1)$$

for some $k \geq \theta$, and $r \in (0,1)$

- 'Q' here stands for 'quotient' of the norms as shown above

R-convergence assuming Lipschitz continuity

- Consider $v^k = \frac{\left\| x^{(0)} - x^* \right\|^2}{2tk} = \frac{\alpha}{k}$, where α is a constant
- Here, we have $\frac{\left\| v^{k+1} - v^* \right\|}{\left\| v^k - v^* \right\|} \leq \frac{K}{K+1}$, where K is the final number of iterations
  - $\frac{K}{K+1} < 1$, but we don't have $\frac{K}{K+1} < r$
- Thus, $v^k = \frac{\alpha}{k}$ is not Q-linearly convergent as there exist no $v < 1$ s.t. $\frac{\alpha/(k+1)}{\alpha/k} = \frac{k}{k+1} \leq v, \forall k \geq \theta$
- Strictly speaking, for Lipschitz continuity alone, gradient descent is not guaranteed to give R-linear convergence
- In practice, Lipschitz continuity gives "almost" R-linear convergence – not too bad!