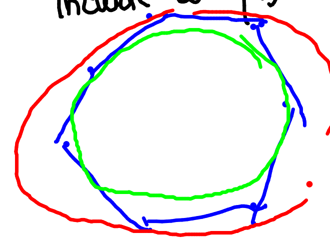
Convex optimization problem

Eg: Given a set of pts, find the tightest enclosing/enclosed ellipsoid.

Enclosing should include all pts



Enclosed ellipsoid should be included within the hull of these pts

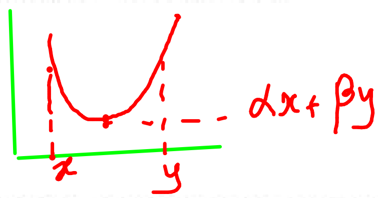
$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & \underbrace{f_i(x) \le b_i}_{x \in C}, \quad i = 1, \ldots, m
\end{array}
$$

- objective and constraint functions are convex:

Initial part of course

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

if $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$



- includes least-squares problems and linear programs as special cases

Convex analysis: Calculus of inequalities

Convex geometry is easiest of geometries

Convex optimisation: Application of convex analysis

Given: $A = \underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}}_{k \text{ columns}} \Big\} n \text{ rows}$

$b = \begin{bmatrix} \quad \end{bmatrix} \Big\} n \text{ rows}$

Need: $Ax \approx b$ ie Ax to be as close to b as possible

ie $\|Ax - b\|^2$ be minimized

$Ax = \begin{bmatrix} \sum_j a_{ij} x_j \end{bmatrix} \rightarrow$ i^{th} component of Ax

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & \cdots & x_k \end{matrix} \\ \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_k \\ \vdots & \vdots & & \vdots \end{bmatrix} \end{array} \rightarrow \text{linear combination } (x_1 \ldots x_k) \text{ of columns of } A$$

V = vector space $\langle V, (S, \cdot), + \rangle$

scalar

e.g. $\mathbb{R}$, Complex

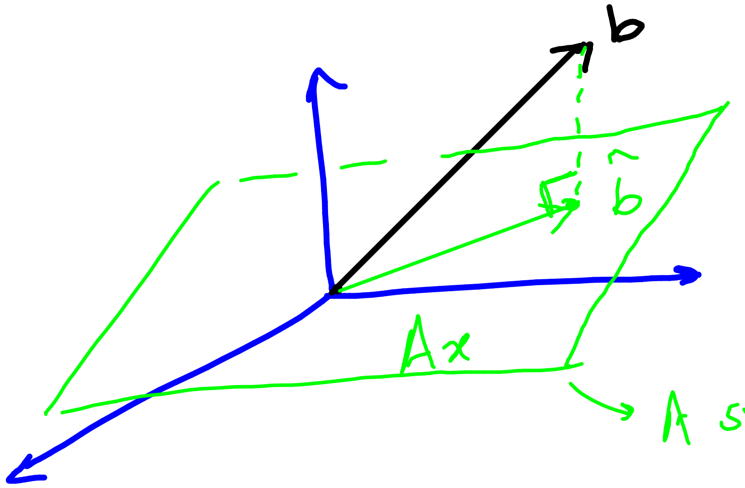
$\forall\, v_1, v_2 \in V$ & $s_1, s_2 \in S$

$s_1 \cdot v_1 + s_2 \cdot v_2 \in V$

$V' \subseteq V$ forms a "subspace" if V' is a vector space ie

$\forall\, s_1, s_2 \in S$ & $v_1', v_2' \in V'$

$s_1 \cdot v_1' + s_2 \cdot v_2' \in V'$



$\langle \mathbb{R}^n, (\mathbb{R}, \cdot), + \rangle$

$\downarrow$ V $\quad \downarrow$ S

$$b \in \mathbb{R}^n$$

For all $x \in \mathbb{R}^k$ $\quad Ax \in \mathbb{R}^n$

vector of scalars from $S = \mathbb{R}$

If $k \leq n$ then Ax is a subspace of $\mathbb{R}^n$

from figure: $b - \hat{b} \perp \hat{b}$

$\forall y \in \mathbb{R}^k$ $\begin{cases} \text{ie } \langle b - \hat{b}, Ay \rangle = 0 \\ \text{ie } \langle b - Ax, Ay \rangle = 0 \end{cases}$

} inner products assumed. We need that $b - \hat{b}$ is orthogonal to subspace Ay ($y \in \mathbb{R}^k$) as necessary & sufficient condition

ie $(Ay)^T (b - Ax) = 0$

ie $y^T A^T (b - Ax) = 0 \quad \forall y \in \mathbb{R}^k$

$\iff A^T (b - Ax) = 0 \iff A^T A x = A^T b$

$\iff x = (A^T A)^{-1} A^T b$

Assumes $A^T A$ is invertible ie A is full column rank

2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

$\theta_1 x_1 + \theta_2 x_2$

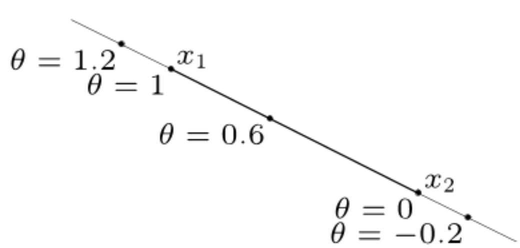
line through x_1, x_2: all points

[Affine] → 1 $\theta_1 + \theta_2 = 1$

$$x = \theta x_1 + (1-\theta)x_2 \qquad (\theta \in \mathbf{R})$$

2 $\theta_1 \geq 0 \quad \theta_2 \geq 0$

3 $\theta_1 + \theta_2 = 1$
& $\theta_1, \theta_2 \geq 0$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$ → insight from linear algebra on geometry etc?

(conversely, every affine set can be expressed as solution set of system of linear equations)

More appropriate name when x_1 & x_2 are pts in real, finite dimensional Euclidean vector space $\mathbb{R}^n$ or $\mathbb{R}^{m \times n}$

Convex set

line segment between x_1 and x_2: all points

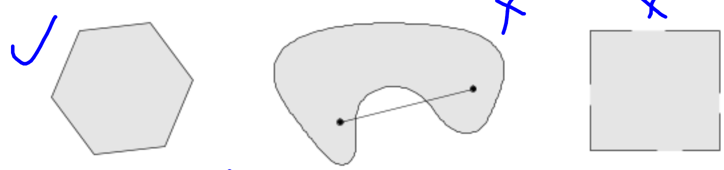
$$x = \theta x_1 + (1-\theta)x_2$$

with $0 \le \theta \le 1$

convex set: contains line segment between any two points in the set

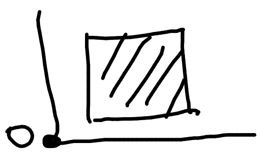
$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



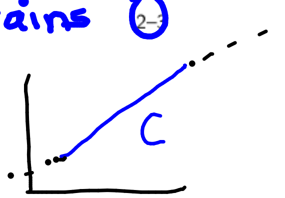
Aside: Convex set is connected: https://en.wikipedia.org/wiki/Connected_space

convex set can, but not necessarily contains '0'



$\mathrm{aff}(C) \ni 0$

$0 \notin \mathrm{aff}(C)$



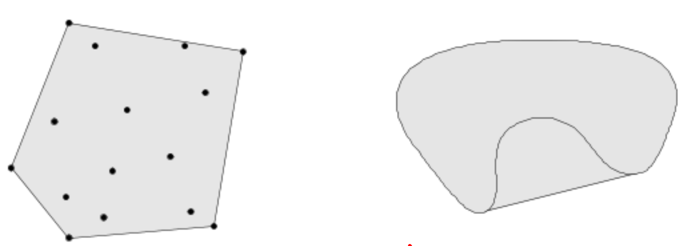
Convex combination and convex hull

convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k = \mathrm{conv}(\{x_1, x_2 \ldots x_k\})$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \ge 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S



conv(S) is always convex