∴ Dual optimization prob is always a concave max/convex minimization problem

H/W: Prove

Lagrange dual function

$\max_{\lambda \succeq 0, \nu} g(\lambda,\nu) = \min_{\lambda \succeq 0, \nu} -g(\lambda,\nu)$

convex function in λ & ν

Lagrange dual function: $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$,

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) = \Omega\left(f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x)\right) \end{aligned}$$

g is concave, can be $-\infty$ for some λ, ν

lower bound property: if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^\star$

($\lambda \succeq 0$ by default means $\lambda \in \mathbb{R}^n_+$)

proof: if $\tilde{x}$ is feasible and $\lambda \succeq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

minimizing over all feasible $\tilde{x}$ gives $p^\star \geq g(\lambda, \nu)$

Least-norm solution of linear equations

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = b \end{array}$$

→ Quadratic in x

dual function

- Lagrangian is $L(x, \nu) = x^T x + \nu^T (Ax - b)$
- to minimize L over x, set gradient equal to zero: → Using necessary condition.

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \quad \Longrightarrow \quad x = -(1/2) A^T \nu$$

- plug in in L to obtain g:

$$g(\nu) = L((-1/2)A^T \nu, \nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu$$

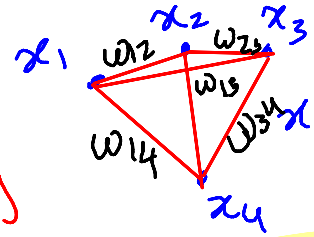
a concave function of ν

Quadratic in ν

lower bound property: $p^\star \geq -(1/4)\nu^T A A^T \nu - b^T \nu$ for all ν

$W = \begin{bmatrix} w_{11} & w_{12} & \\ w_{21} & \ddots & \\ & & w_{44} \end{bmatrix}$

Two-way partitioning



$x_i \in \{-1, +1\}$

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & \underbrace{x_i^2 = 1}_{\text{Non convex domain}}, \quad i = 1, \ldots, n \end{array}$$

→ energy level of config

Ising Models

- a nonconvex problem; feasible set contains 2^n discrete points
- interpretation: partition $\{1, \ldots, n\}$ in two sets; W_{ij} is cost of assigning i, j to the same set; $-W_{ij}$ is cost of assigning to different sets

dual function

$$\begin{aligned} g(\nu) = \inf_x (x^T W x + \sum_i \nu_i (x_i^2 - 1)) &= \inf_x x^T (W + \mathbf{diag}(\nu)) x - \mathbf{1}^T \nu \\ &= \begin{cases} -\mathbf{1}^T \nu & W + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

p.s.d constraint

lower bound property: $p^\star \geq -\mathbf{1}^T \nu$ if $W + \mathbf{diag}(\nu) \succeq 0$

example: $\nu = -\lambda_{\min}(W)\mathbf{1}$ gives bound $p^\star \geq n\lambda_{\min}(W)$

$W + \begin{bmatrix} -\lambda_{\min} & 0 \\ 0 & -\lambda_{\min} \end{bmatrix} \succeq 0$ ← substitute

Lagrange dual and conjugate function

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & \underbrace{Ax \preceq b}_{\lambda}, \quad \underbrace{Cx = d}_{\nu} \end{array}$$

dual function

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbf{dom}\, f_0} \left(f_0(x) + (A^T \lambda + C^T \nu)^T x - b^T \lambda - d^T \nu \right) \\ &= -f_0^*(-A^T \lambda - C^T \nu) - b^T \lambda - d^T \nu \end{aligned}$$

- recall definition of conjugate $f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$
- simplifies derivation of dual if conjugate of f_0 is kown

example: entropy maximization

$$f_0(x) = \sum_{i=1}^n x_i \log x_i, \qquad f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

} constraints are linear (probability) constraints

Reminds you of Entropy classifier Logistic regression

The dual problem

Lagrange dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- finds best lower bound on $p^\star$, obtained from Lagrange dual function
- a convex optimization problem; optimal value denoted $d^\star$
- λ, ν are dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \mathbf{dom}\, g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom}\, g$ explicit

example: standard form LP and its dual (page 5–5) (also seen for conic Prog CP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array} \equiv \begin{array}{ll} \text{maximize} & -b^T \nu \\ \text{subject to} & A^T \nu + c \succeq 0 \end{array}$$

If both primal & dual are feasible & one of them is strictly feasible $\Rightarrow$ zero duality (strong duality)

For detailed treatment on duality for LP & CLP see
http://www.cse.iitb.ac.in/~cs709/notes/enotes/lecture24b.pdf

Weak and strong duality

weak duality: $d^\star \leq p^\star$

- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems

  for example, solving the SDP

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}^T \nu \\ \text{subject to} & W + \mathbf{diag}(\nu) \succeq 0 \end{array}$$

  gives a lower bound for the two-way partitioning problem on page 5–7

strong duality: $d^\star = p^\star$

- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called **constraint qualifications**

Approach ① for zero duality

Slater's constraint qualification

Again seen in strong conic duality

strong duality holds for a convex problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

if it is strictly feasible, *i.e.*,

$$\exists x \in \mathbf{int}\, \mathcal{D}: \qquad f_i(x) < 0, \quad i = 1, \ldots, m, \qquad Ax = b$$

- also guarantees that the dual optimum is attained (if $p^\star > -\infty$)
- can be sharpened: *e.g.*, can replace $\mathbf{int}\, \mathcal{D}$ with $\mathbf{relint}\, \mathcal{D}$ (interior relative to affine hull); linear inequalities do not need to hold with strict inequality, . . .
- there exist many other types of constraint qualifications

Inequality form LP

primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

dual function

$$g(\lambda) = \inf_x \left((c + A^T\lambda)^T x - b^T \lambda \right) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0, \quad \lambda \succeq 0 \end{array}$$

- from Slater's condition: $p^\star = d^\star$ if $A\tilde{x} \prec b$ for some $\tilde{x}$
- in fact, $p^\star = d^\star$ except when primal and dual are infeasible

OR

Slater's condition: Sufficient for LP & CLP.
For LP/CLP simply feasibility of primal & dual is sufficient

Approach ② for zero duality

Complementary slackness

assume strong duality holds, $x^\star$ is primal optimal, $(\lambda^\star, \nu^\star)$ is dual optimal

$$\begin{aligned} f_0(x^\star) = g(\lambda^\star, \nu^\star) &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) + \sum_{i=1}^p \nu_i^\star h_i(x) \right) \\ &\le f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) + \sum_{i=1}^p \nu_i^\star h_i(x^\star) \\ &\le f_0(x^\star) \end{aligned}$$

$h_j(x^*) = 0$

$\lambda_i^* \ge 0$ $\quad f_i(x^*) \le 0$

$\because \lambda_i^*$ is dual feasible $\quad \because x^*$ is primal feasible

hence, the two inequalities hold with equality

Necessary conditions for zero duality

- $x^\star$ minimizes $L(x, \lambda^\star, \nu^\star)$
- $\lambda_i^\star f_i(x^\star) = 0$ for $i = 1, \ldots, m$ (known as complementary slackness):

$$\lambda_i^\star > 0 \Longrightarrow f_i(x^\star) = 0, \qquad f_i(x^\star) < 0 \Longrightarrow \lambda_i^\star = 0$$

Karush-Kuhn-Tucker (KKT) conditions

the following four conditions are called KKT conditions (for a problem with differentiable f_i, h_i):

1. primal constraints: $f_i(x) \le 0$, $i = 1, \ldots, m$, $h_i(x) = 0$, $i = 1, \ldots, p$
2. dual constraints: $\lambda \succeq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0$, $i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^m \lambda_i \nabla f_i(x) + \sum_{i=1}^p \nu_i \nabla h_i(x) = 0$$

from page 5–17: if strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions

KKT conditions for convex problem

if $\tilde{x}$, $\tilde{\lambda}$, $\tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

"only if" assumes differentiability of f, g_i & h_j

if **Slater's condition** is satisfied:

x is optimal if and only if there exist λ, ν that satisfy KKT conditions

- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

example: water-filling (assume $\alpha_i > 0$)

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \log(x_i + \alpha_i) \\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1 \end{array}$$

x is optimal iff $x \succeq 0$, $\mathbf{1}^T x = 1$, and there exist $\lambda \in \mathbf{R}^n$, $\nu \in \mathbf{R}$ such that

$$\lambda \succeq 0, \qquad \lambda_i x_i = 0, \qquad \frac{1}{x_i + \alpha_i} + \lambda_i = \nu$$

- if $\nu < 1/\alpha_i$: $\lambda_i = 0$ and $x_i = 1/\nu - \alpha_i$
- if $\nu \geq 1/\alpha_i$: $\lambda_i = \nu - 1/\alpha_i$ and $x_i = 0$
- determine ν from $\mathbf{1}^T x = \sum_{i=1}^n \max\{0, 1/\nu - \alpha_i\} = 1$

interpretation

- n patches; level of patch i is at height α_i
- flood area with unit amount of water
- resulting level is $1/\nu^\star$