② In fact, if K is a proper cone then K^* is also proper

③ A more general notation for CP

CP: $\min_{x} \; c^T x$ $\geq$ CD: $\max_{\lambda} \; \langle b, \lambda \rangle$

s.t $Ax - b \in K$, $x \in \mathbb{R}^n$ s.t $A^*\lambda = c$, $\lambda \in K^*$

$A: \mathbb{R}^n \to V$ is a linear map

(Eg: V could be S^n
$K \subseteq V$ could be S^n_+)

$A^*: V \to \mathbb{R}^n$ is the adjoint linear map of $A: \mathbb{R}^n \to V$

ie $\langle Au, v \rangle_V = \langle u, A^* v \rangle_{\mathbb{R}^n}$
$\forall u \in \mathbb{R}^n$ & $\forall v \in V$

H/W: Could the implications we will derive hold if $\mathbb{R}^n$ is further generalized

Eg: if A is matrix multiplication then $A^* = A^T$ (verify)

④

$$\min_{\lambda \in K^*} \langle \lambda, x \rangle = \begin{cases} 0 & \text{if } x \in K \text{, since } 0 \in K^* \\ -\infty & \text{if } x \notin K \end{cases}$$

Using this result, we can get the conic program as the dual of its dual (assuming $K^{**} = K$)

(5) Consider Conic dual

$$d^* = \max_{\lambda} \langle b, \lambda \rangle$$

$$A^*\lambda = c, \ \lambda \in K^*$$

$$d^* = \max_{\lambda} \min_{\substack{y \in \mathbb{R}^n \\ x \in K^{**} = K}} \langle b, \lambda \rangle + \langle y, (c - A^*\lambda) \rangle + \underbrace{\langle x, \lambda \rangle}_{\text{from pt (4), } \min_{x \in K^*} \langle x, \lambda \rangle = 0 \text{ if } \lambda \in K^*}$$

λ s.t. $\lambda \in K^*$, $A^*\lambda = c$

$$\leq \max_{\lambda} \min_{\substack{y \in \mathbb{R}^n \\ x \in K}} \langle b, \lambda \rangle + \langle y, (c - A^*\lambda) \rangle + \langle x, \lambda \rangle$$

[Max min inequality]

$$\min_y f(x,y) \leq f(x,y)$$

$$\Rightarrow \max_x \min_y f(x,y) \leq \max_x f(x,y) \quad (\forall y)$$

$$\Rightarrow \max_x \min_y f(x,y) \leq \min_y \max_x f(x,y)$$

$$\leq \min_{\substack{y \in \mathbb{R}^n \\ x \in K}} \underbrace{\max_{\lambda} \langle b, \lambda \rangle + \langle y, (c - A^*\lambda) \rangle + \langle x, \lambda \rangle}$$

$$\max_{\lambda} \underbrace{\langle b - Ay + x, \lambda \rangle} + \langle y, c \rangle \begin{cases} = \langle y, c \rangle & \text{if } b - Ay + x = 0 \text{ i.e if } Ay - b \in K \\ = \infty & \text{o/w} \end{cases}$$

$\because \langle y, A^*\lambda \rangle = \langle Ay, \lambda \rangle$ $\forall y \in \mathbb{R}^n$ & $\lambda \in V$

$$= \min_{y \in \mathbb{R}^n} \langle y, c \rangle \quad \text{s.t } Ay - b \in K$$

Dual cones and generalized inequalities

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbf{R}^n_+$: $K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

dual cones of proper cones are proper, hence define generalized inequalities:

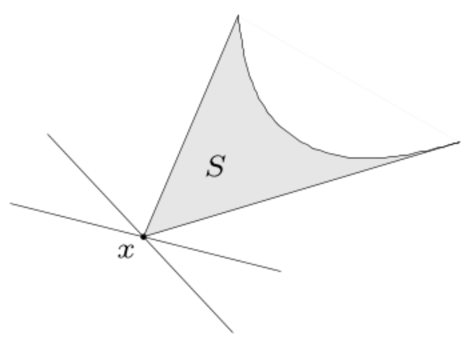
$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

ie $\forall\, y \in S,\ x \preceq_K y$

Minimum and minimal elements via dual inequalities

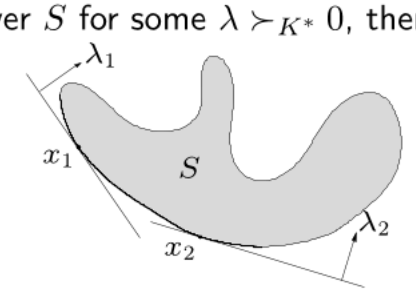
minimum element w.r.t. $\preceq_K$

x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S



minimal element w.r.t. $\preceq_K$

- if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$, then x is minimal



- if x is a minimal element of a *convex* set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

FROM DUAL OF NORM CONE TO DUAL NORM

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$

The dual of $K = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \le t\}$

is $K^* = \{(u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \le v\}$

Where $\|u\|_* = \sup\{u^T x \mid \|x\| \le 1\}$

Proof: We need to show that

$$x^T u + tv \ge 0 \text{ whenever } \|x\| \le t \iff \|u\|_* \le v. \tag{2.20}$$

1) Let us start by showing that the righthand condition on (u, v) implies the lefthand condition. Suppose $\|u\|_* \le v$, and $\|x\| \le t$ for some $t > 0$. (If $t = 0$, x must be zero, so obviously $u^T x + vt \ge 0$.) Applying the definition of the dual norm, and the fact that $\|-x/t\| \le 1$, we have

$$u^T(-x/t) \le \|u\|_* \le v,$$

and therefore $u^T x + vt \ge 0$.

2) Next we show that the lefthand condition in (2.20) implies the righthand condition in (2.20). Suppose $\|u\|_* > v$, *i.e.*, that the righthand condition does not hold. Then by the definition of the dual norm, there exists an x with $\|x\| \le 1$ and $x^T u > v$. Taking $t = 1$, we have

$$u^T(-x) + v < 0,$$

which contradicts the lefthand condition in (2.20).

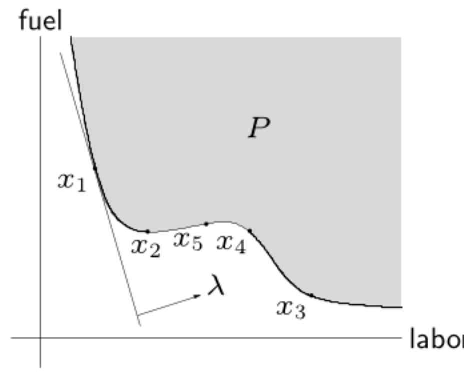
(Proof from Boyd)

optimal production frontier

- different production methods use different amounts of resources $x \in \mathbf{R}^n$
- production set P: resource vectors x for all possible production methods
- efficient (Pareto optimal) methods correspond to resource vectors x that are minimal w.r.t. $\mathbf{R}^n_+$

example $(n = 2)$

x_1, x_2, x_3 are efficient; x_4, x_5 are not



① LP is only a special case of CP with $K = \boxed{\mathbb{R}^+_n}$

② When will solution to CP = solution to CD?

CP: $\min_{x \in \mathbb{R}^n} \langle c, x \rangle$ s.t. $Ax \geq_K b$

CD: $\max_{\lambda \in K^*} \langle b, \lambda \rangle$ s.t. $A^* \lambda = c$

Since $K^{**} = K$, we saw that dual of CD is CI.

While there exist multiple ways of writing CP & CD, hereafter we pick another standard format (to help you get used to various representations)

CP: $\min \langle c, x \rangle_V$

s.t. $Ax = b$, $x \in K \subseteq V$

F_p $\quad A: V \to \mathbb{R}^n$

CD: $\max \langle b, \lambda \rangle_{\mathbb{R}^n}$

s.t. $c - \lambda^T A \in K^*$, $\lambda \in \mathbb{R}^n$

F_d $\quad K^* \subseteq V$

Examples:

① LP

$$\text{LP:} \quad \min_x \; c^T x \quad \text{s.t.} \; Ax = b, \; x \geq 0$$

$$\text{Dual:} \quad \max_\lambda \; \lambda^T b \quad \text{s.t.} \; c - \lambda^T A \in \mathbb{R}^n_+, \; \lambda \in \mathbb{R}^n$$

② Semi-definite program

$$\text{SDP:} \quad \inf_X \; \langle C, X \rangle_F \quad (i=1..m) \; \text{s.t.} \; \langle A_i \cdot X \rangle = b_i, \quad A_i \in S^n, \; X \in S^n_+$$

$$\text{SDD} \quad \sup_\lambda \; \lambda^T b \quad \text{s.t.} \; C - \sum_{i=1}^m \lambda^T A_i \in S_n^+, \; \lambda \in \mathbb{R}^m$$

Weak duality theorem for CP

$\langle c, x \rangle_V \geq \langle b, \lambda \rangle_{\mathbb{R}^n}$ where $x \in F_p$ $\lambda \in F_d$

If $V = \mathbb{R}^n$ & $V_b = \mathbb{R}^m$ & $\langle , \rangle$ is dot product

$$c^T x - b^T \lambda = \underbrace{(c - A^T\lambda)^T x}\geq 0$$

(we have proved this twice: from CP → CD & from CD → CP)

$\langle c, x \rangle_V - \langle b, \lambda \rangle_{\mathbb{R}^n}$ is called duality gap

Corollary ① If CP or CD is feasible but unbounded, then the other is infeasible or has no feasible soln

Eg: min x s.t $x \in (0, 10]$

② If a pair of feasible solutions can be found to the primal & dual problems with equal objective value then they are both optimal

Proof: Let $\langle c, x^* \rangle_V = \langle b, \lambda^* \rangle$ → I

s t $x^* \in F_p$ & $\lambda^* \in F_d$

But we know: $\langle c, x^* \rangle \geq \min_{x \in F_p} \langle c, x \rangle \geq \max_{x \in F_d} \langle b, \lambda \rangle \geq \langle b, \lambda^* \rangle$ → II

Because of I, all inequalities should become equalities in II

$\Rightarrow \min_{x \in F_p} \langle c, x \rangle = \max_{x \in F_d} \langle b, \lambda \rangle$

Q: Does "STRONG DUALITY" hold for CP?

when?

(a) $C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $K = S^3_+$

in an SDP?

(b) $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$K \in S^2_+$

in an SDP?

STRONG DUALITY THM:

1. Let CP or CD be infeasible & let other be feasible & have an interior. Then the other is unbounded

2. Let CP and CD be both feasible, and let one of them have an interior. Then there is 0 duality gap

3. Let CP and CD be both feasible and have interior. Then both have optimal solutions with 0 duality gap

Proof:

We need the theorem of alternatives to prove strong conic duality [Also called Farkas' Lemma for Convex Cone]

Theorem of alternatives:

Consider $\{x \mid Ax=b,\ x\in K\}$ for a proper cone $K \subseteq V$ & $A: V \to \mathbb{R}^n$

Suppose $\exists\ \lambda$ s.t $-A^*\lambda \in \text{int}(K^*)$. Then

$$\boxed{\begin{array}{l} \text{(a)}\ \{x \mid Ax=b,\ x\in K\} \text{ has a feasible soln } x \quad \text{iff} \\ \text{(b)}\ \{\lambda \mid -A^*\lambda \in K^*, \langle b,\lambda\rangle > 0\} \text{ has no feasible solution} \end{array}}$$

PROOF:

(i) $C = \{y = Ax \in \mathbb{R}^m, x\in K\}$ is a closed convex set

(ii) Let $\bar{\lambda}$ be s.t $-A^*\bar{\lambda} \in K^*$ and Let $\{x \mid Ax=b, x\in K\}$ have a feasible solution $\bar{x}$

$\Rightarrow -\langle\bar{\lambda}, b\rangle = -\langle\bar{\lambda}, A\bar{x}\rangle$ . . . .

. . . . ie $\{\lambda \mid -A^*\lambda \in K, \langle b,\lambda\rangle > 0\}$ has no solution

(iii) Let $\{x \mid Ax=b, x\in K\}$ have no feasible solution ie b [illegible] C

We will show that $\{\lambda \mid -A^*\lambda \in K^*, \langle \lambda, b\rangle > 0\}$ must be non-empty

Since C is a closed convex set, from the strict separating hyperplane theorem, $\exists\, \lambda \in \mathbb{R}^m$ s.t

$$> \langle \lambda, y\rangle \quad \forall\, y \in C$$

Since $\exists\, x \in K$ s.t $Ax = y$ for any $y \in C$

$$> \langle \lambda, Ax\rangle = \qquad \forall\, x \in K$$

- Thus, $\langle A^*\lambda, x\rangle$ is bounded above $\forall\, x \in K$
- Since $0 \in K$, > 0
- Additionally, it must be that $\langle A^*\lambda, x\rangle \leq 0$ $\forall x \in K$.

  Otherwise if $\exists\, x \in K$ s.t $\langle A^*\lambda, x\rangle > 0$ then if $\alpha \to +\infty$ then $\to \infty$ contradicting that is bounded above for all x
- Since $\langle A^*\lambda, x\rangle \leq 0$ $\forall\, x$

$$\geq 0 \quad \forall\, x \Rightarrow \quad \in K^*.$$

- Thus, λ is a (feasible) solution for - - - .

- - - complete the proof - - - -