2 Equivalent definitions of affine sets:

① $\forall\ x_1, x_2 \in S \qquad \theta_1 x_1 + \theta_2 x_2 \in S \qquad \theta_1 + \theta_2 = 1$

② $\{x \mid Ax = b\}$ for some $m \times n$ matrix A

<u>Proof</u>: ② $\Rightarrow$ ① is trivial since $Ax_1 = b$ & $Ax_2 = b$

$\Rightarrow A(\theta_1 x_1 + \theta_2 x_2) = b$ if $\theta_1 + \theta_2 = 1$

① $\Rightarrow$ ② . . . suggestion: Subtract "p" $\in S$ from S

ie $S_p = S - p$ & show S_p is a v.s

For answer: pages 145 to 181 of

http://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

Rank of A

r=m=n	r=m<n or r≤m<n	r=n<m
R=I	R=[I F]	R=[I 0]^T
Unique solution	Infinitely many solutions	0 or 1 solution

Affine set = Point | shifted v.s | empty or point

Figure 3.3: Summary of the properties of the solutions to the system of equations $Ax = b$.

$A_{m\times n}$ matrix

$A\,x_{particular} = b$

$A\,x_{nullspace} = 0$

(x_2, x_4) Assign some values to free variables

$$A\mathbf{x}_{complete} = A(\mathbf{x}_{particular} + \mathbf{x}_{nullspace}) = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Example:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

$\Downarrow$ (Gauss Elimination)

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \overset{E_{2,1},E3,1}{\Longrightarrow} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\overset{E3,2}{\Longrightarrow} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

$r = 2 \quad m = 3 \quad n = 4$

No solution is a possibility { condition for solvability: $b_3 - b_1 - b_2 = 0$

Thus:

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable when $\mathbf{b}$ is in the column space $C(A)$.

Another way of describing solvability is:

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable if a combination of the rows of A produces a zero row, the requirement on $\mathbf{b}$ is that the same combination of the components of $\mathbf{b}$ has to yield zero.

Steps to find $x_{particular}$:

1. $\mathbf{x}_{particular}$[2]: Set all free variables (corresponding to columns with no pivots) to 0. In the example above, we should set $x_2 = 0$ and $x_4 = 0$.
2. Solve $A\mathbf{x} = \mathbf{b}$ for pivot variables.

In this example:

$$x_1 + 2x_3 = b_1 \quad \& \quad 2x_3 = b_2 - 2b_1$$

$$\Rightarrow \quad \mathbf{x}_{particular} = \begin{bmatrix} b_2 + 3b_1 \\ 0 \\ \frac{b_2 - 2b_1}{2} \\ 0 \end{bmatrix}$$

Now: $\mathbf{x}_{complete} = \mathbf{x}_{particular} + \mathbf{x}_{nullspace}$ since $x_{nullspace}$ is s.t $Ax_{nullspace} = 0$

$$\boxed{A\mathbf{x}_{complete} = A(\mathbf{x}_{particular} + \mathbf{x}_{nullspace}) = \mathbf{b} + \mathbf{0} = \mathbf{b}}$$

Eg: if we choose $\mathbf{b} = [5\ 1\ 6]^T$, we get $\mathbf{x}_{particular} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$ &

$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{x_{nullspace}} \tag{3.36}$$

$x_{particular}$

Show that $x_{complete} = \theta x_1 + (1-\theta)x_2$ for some $x_1, x_2 \in \mathbb{R}^4$ & $\theta \in \mathbb{R}$

$\xrightarrow[\text{that}]{\text{Proves}}$ $\{x \mid Ax = b\}$ is an affine set

Q: What is a more generalised definition of affine sets?

Procedure to obtain A & b given an affine set S

① Let $p \in S$ $\quad S-p$

Then claim: $\{x-p \mid x \in S\}$ is a vector space

Call it S_p

H/W

② Identify A s.t $\forall x \in S_p, \; Ax = 0$

i.e rows of A could form basis of $S_p^{\perp}$

Basically $A x_{nullspace} = 0$

③ Identify $b = Ap$

Basically the $x_{particular}$ giving you $A x_{particular} = b$ in prev example

More appropriate name when x_1 & x_2 are pts in real, finite dimensional Eucledian vector space $\mathbb{R}^n$ or $\mathbb{R}^{m \times n}$

Convex set

line segment between x_1 and x_2: all points

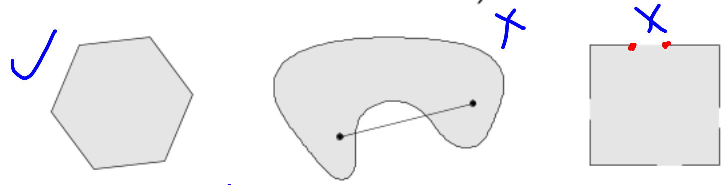
$$x = \theta x_1 + (1 - \theta) x_2$$

with $0 \le \theta \le 1$

convex set: contains line segment between any two points in the set

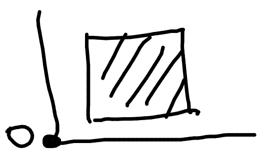
$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta) x_2 \in C$$

examples (one convex, two nonconvex sets)



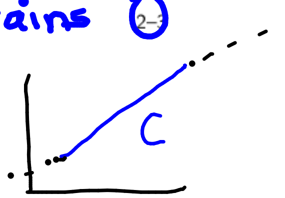
Aside: Convex set is connected: https://en.wikipedia.org/wiki/Connected_space

convex set can, but not necessarily contains '0'



$aff(C) \ni 0$

$0 \notin aff(C)$



Convex combination and convex hull

convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k = \mathrm{conv}(\{x_1, x_2 \ldots x_k\})$$

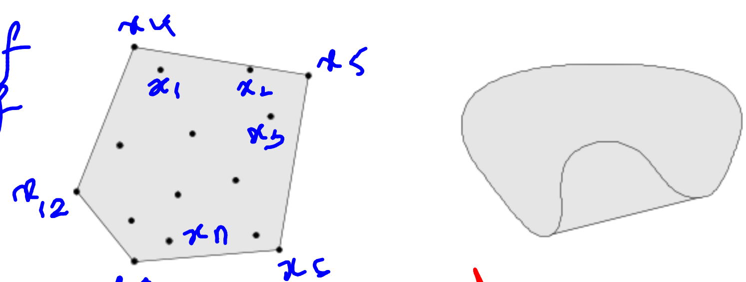
with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \ge 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S

Convex hull of a finite set of pts is a polyhedron

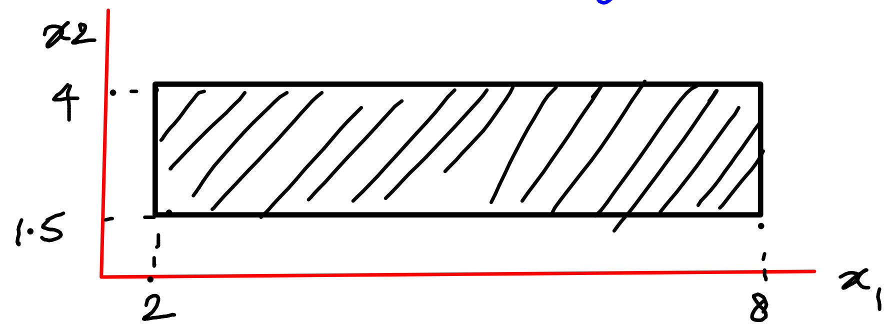
$Ax \geq b$

$\{x \mid a_1^T x \geq b_1 \ldots a_n^T x \geq b_n\}$



conv(S) is always convex

$Ax \geq b$ for a rectangle



$$\left.\begin{array}{ll} x_1 \geq 2 & x_2 \geq 1.5 \\ -x_1 \geq -8 & -x_2 \geq -4 \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -8 \\ 1.5 \\ -4 \end{bmatrix}$$

Convex cone

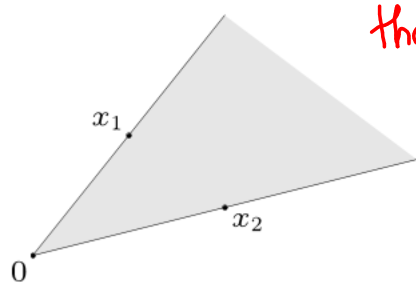
Cone: C is a cone if $\forall x \in C$, $\theta x \in C$ for $\theta \geq 0$

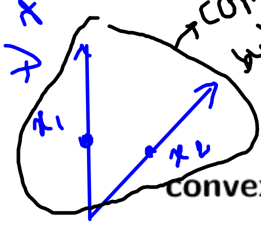
conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

If $\theta_1 = 0$ & $\theta_2 = 0$ then $x = 0 \in$ Convex cone





cone but not convex

convex cone: set that contains all conic combinations of points in the set

Hyperplanes and halfspaces

Defn 1

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

Set of all affine combinations of n points $\{x_1 .. x_n\}$ in $\mathbb{R}^n$ st $\{x_1 ... x_n\}$ are linearly independent

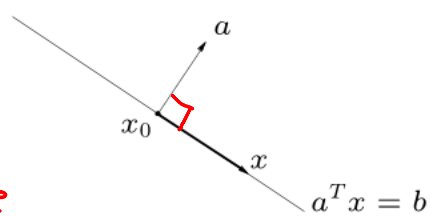
Defn 2

a is normal

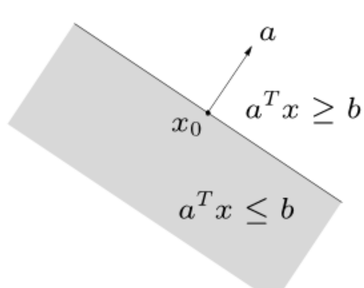
$x_0 \in H_a$

$\{x \mid (x - x_0) \perp a\}$

$= \{x \mid x^T a = x_0^T a = b\}$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

By defn 1

But NOT affine

Q: What is the relation between

A = affine set : $\theta_1 + \theta_2 = 1$

S = convex set : $\theta_1 + \theta_2 = 1 \quad \theta_1, \theta_2 \geq 0$

C = convex cone: $\theta_1, \theta_2 \geq 0$

Every affine set is convex.
Every convex cone is convex.

- Family of affine sets is subset of family of convex sets
- Family of cones is subset of family of convex sets

Thus: (a) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = set of all convex combinations of pts in S

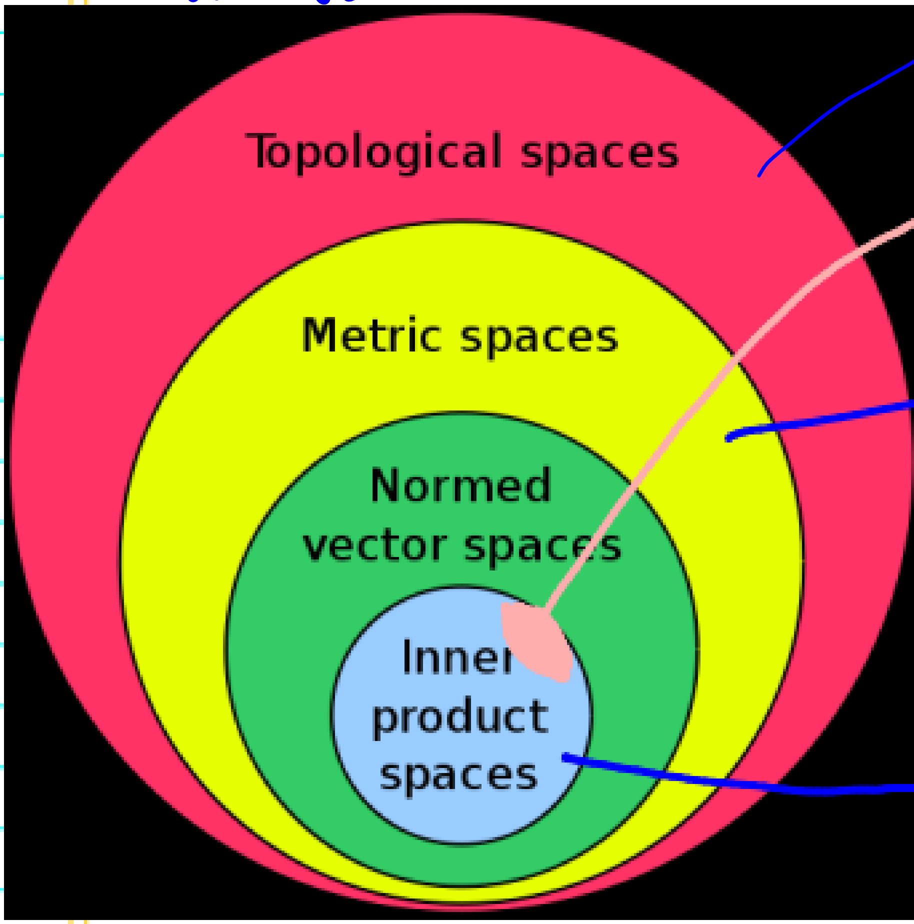
(b) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = Smallest convex set that contains S [Prove as h/w]

Also: The idea of a convex combination can be generalised to include infinite sums, integrals, and, in the most general form, probability distributions

Similarly: (a) $\underbrace{\text{Conic/Affine hull}(S)}_{\text{conic}(S) \text{ or aff}(S)}$ = set of all conic/affine combination of pts in S

(b) $\underbrace{\text{Conic/Affine hull}}_{\text{conic}(S) \text{ or aff}(S)}$ (S) = Smallest conic/affine set that contains S

IN GENERAL



Need nbrhood

Hilbert space

Tringle inequality

$\|v\|^2 = [v, v]$

vector space with an inner prod

Source: http://en.wikipedia.org/wiki/Space_(mathematics)

A hierarchy of mathematical spaces: The inner product induces a norm. The norm induces a metric. The metric induces a topology.

Topological space: Set of points along with a set of neighborhoods of each point, with certain axioms required to be satisfied by the pts & their nbrhoods

Metric space: Set of points with a notion of "distance" between elements $d(x,y)$ must be (a) non-negative (b) $d(x,y)=0$ iff $x=y$ (c) symmetric (d) satisfy triangle inequality

Assuming you have understood vector space

Normed vector space: A vector space on which a norm is defined. (see page number 4 for definition of norm)

Definitions: In topological space, $\{x_i\}$ could converge to a limit $\lim_{i\to\infty} x_i$

- $\lim_{i\to\infty} \frac{1}{i} = 0$

/en.wikipedia.org/wiki/Limit_point

$cl(S)$ when S is a topological space

→ Should consist of S

union with

→ Should consist of $\lim_{i\to\infty} x_i$ for every convergent sequence $\{x_i\} \subseteq S$

For general topological space

S is closed if $cl(S) = S$

S is open if S^c is closed

$$\text{interior}(S) = \bigcup_{\substack{S' = \text{open} \\ S' \subseteq S}} S'$$

$$bnd(S) = cl(S) - int(S)$$

$$\stackrel{?}{=} cl(S) \cap cl(S^c)$$

$$relbnd(S) = cl(S) - relint(S)$$

$$relint(S) = \{x \mid x \in S \text{ s.t } \exists \epsilon > 0 \text{ s.t } \{y \mid \|x-y\| < \epsilon\} \cap aff(S) \subseteq S\}$$

with norm $\|\cdot\|$

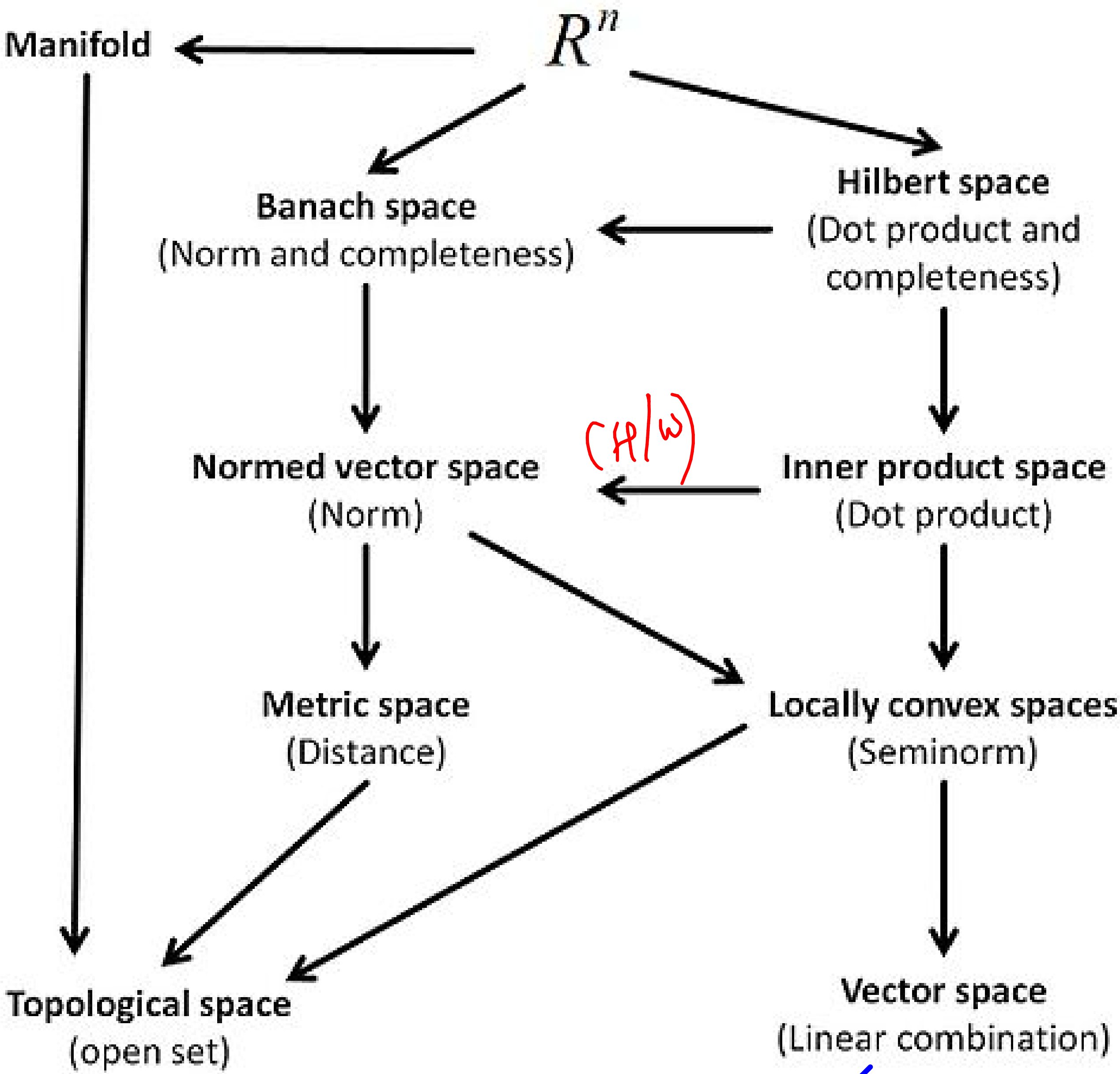
$\forall x \in S, \exists \epsilon > 0$ s.t $\{y \mid \|y-x\| \leq \epsilon\} \subseteq S$ — open set in Normed v.s

$bnd(S) = \partial(S)$

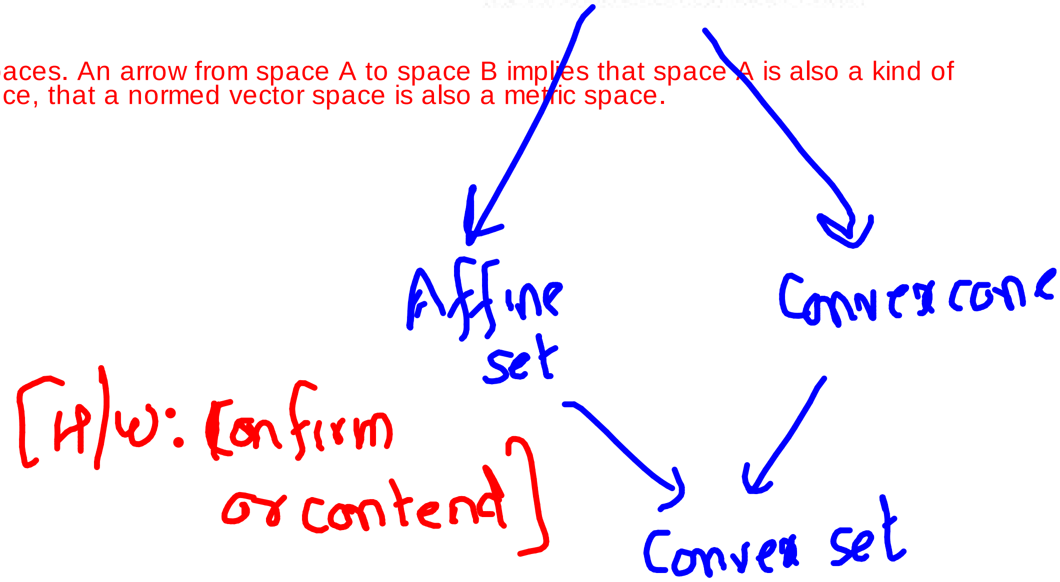
x belongs to the normed space

$$cl(S) = \{x \mid \forall \epsilon > 0 \; S \cap \{y \mid \|x-y\| < \epsilon\} \neq \phi\}$$

$$int(S) = \{x \mid x \in S \text{ s.t } \exists \epsilon > 0 \text{ s.t } \{y \mid \|x-y\| < \epsilon\} \subseteq S\}$$



Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



[H/w: Prove that "normed" space is a "metric" space]

Inner product space: It is a vector space over a field of scalars along with an inner product

eg: $\mathbb{R}$

an algebraic structure with addition, subtraction, multiplication & division

associative & commutative

must be commutative, associative & distributive

multiplicative inverse must exist

ⓐ (Conjugate) symmetry:

$\langle x,y\rangle = \langle y,x\rangle$

ⓑ Linearity in the first argument

$\langle ax,y\rangle = a\langle x,y\rangle$

$\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$

ⓒ Positive definiteness:

$\langle x,x\rangle \geq 0$ with equality iff $x=0$