# Convex cone

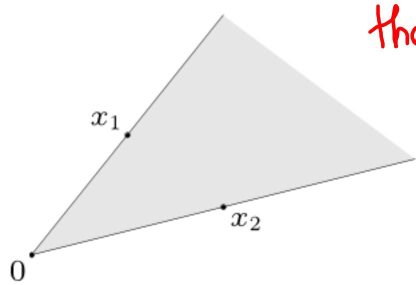
Cone: $C$ is a cone if $\forall x \in C$, $\theta x \in C$ for $\theta \geq 0$

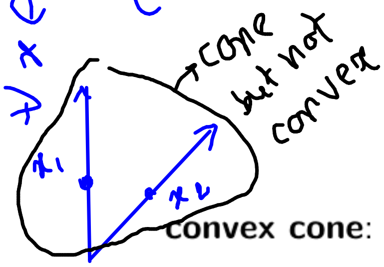
**conic (nonnegative) combination** of $x_1$ and $x_2$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

If $\theta_1 = 0$ & $\theta_2 = 0$ then $x = 0 \in$ Convex Cone




**convex cone**: set that contains all conic combinations of points in the set



# Hyperplanes and halfspaces

Defn 1

Defn 2

**hyperplane**: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

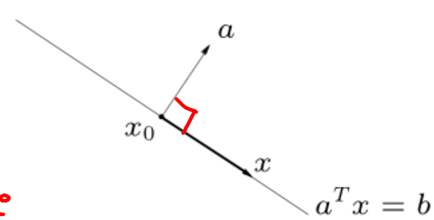
Set of all affine combinations of $n$ points $\{x_1 .. x_n\}$ in $\mathbb{R}^n$ st $\{x_1 ... x_n\}$ are linearly independent

$a$ is normal

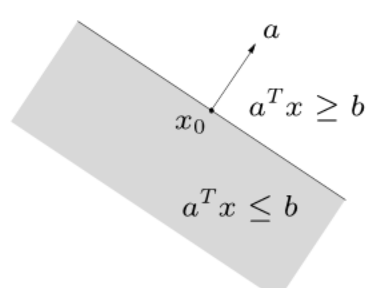
$x_0 \in H_a$

$\{x \mid (x - x_0) \perp a\}$

$= \{x \mid x^T a = x_0^T a = b\}$




**halfspace:** set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$




- $a$ is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

By defn 1

But NOT affine

**Thus:** (a) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = set of all convex combinations of pts in S

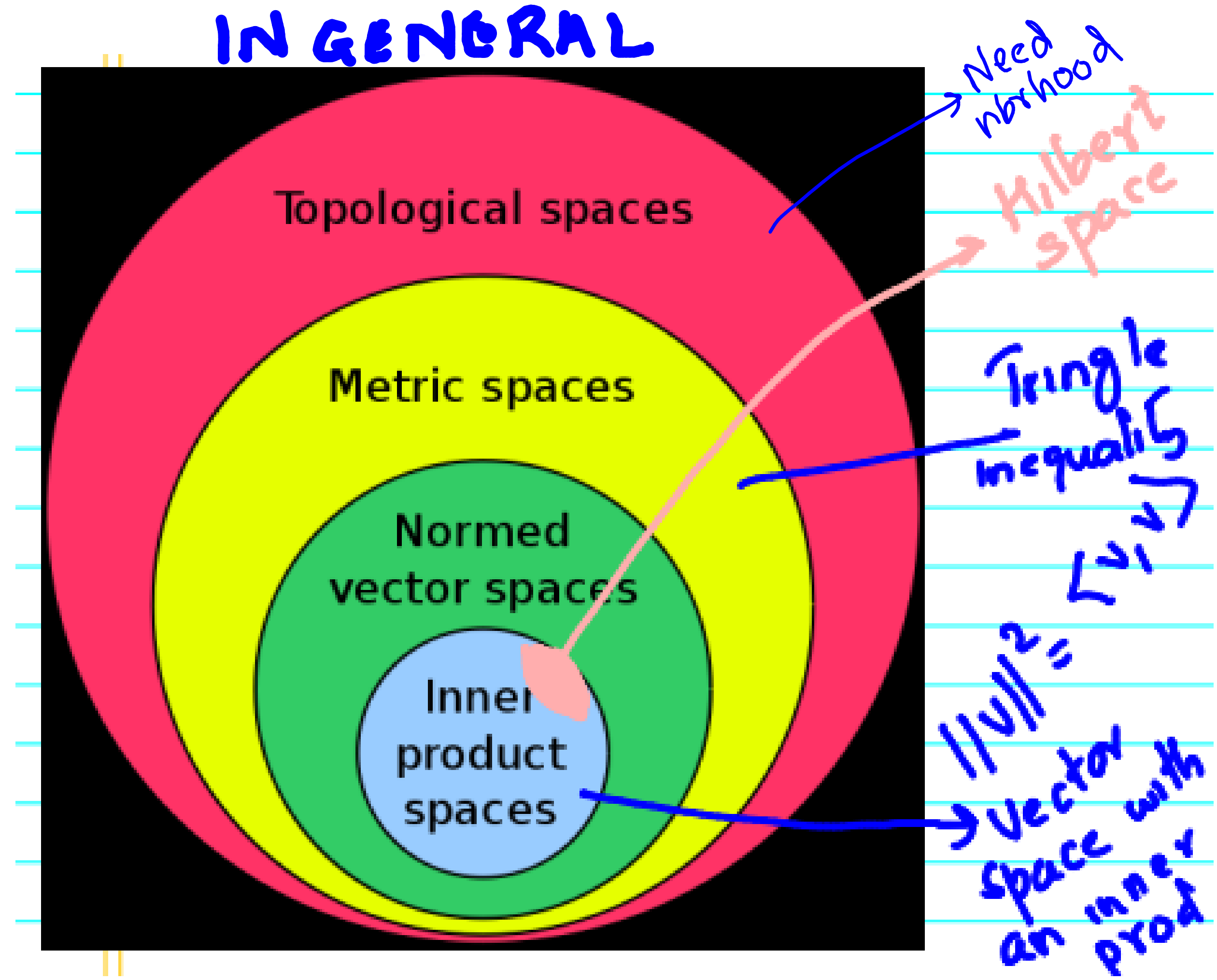
(b) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = smallest convex set that contains S [Prove as h/w]

**Also:** The idea of a convex combination can be generalised to include infinite sums, integrals, and, in the most general form, probability distributions

**Similarly:** (a) $\underbrace{\text{Conic/Affine hull}(S)}_{\text{conic}(S) \text{ or aff}(S)}$ = set of all conic/affine combination of pts in S

(b) $\underbrace{\text{Conic/Affine hull}}_{\text{conic}(S) \text{ or aff}(S)}$ (S) = smallest conic/affine set that contains S

# IN GENERAL




Source: http://en.wikipedia.org/wiki/Space_(mathematics)

A hierarchy of mathematical spaces: The inner product induces a norm. The norm induces a metric. The metric induces a topology.

**Topological space:** Set of points along with a set of neighborhoods of each point, with certain axioms required to be satisfied by the pts & their nbrhoods

**Metric space:** Set of points with a notion of "distance" between elements
$d(x,y)$ must be (a) non-negative (b) $d(x,y)=0$ iff $x=y$ (c) symmetric (d) satisfy triangle inequality

Assuming you have understood vector space

**Normed vector space:** A vector space on which a norm is defined. (see page number 4 for definition of norm)

# Neighborhood axioms for Topological spaces

① Each point $x \in X$ belongs to each of its neighborhoods $N(x)$

② If $N_1 \subseteq X$ & $N_2 \subseteq X$ and $N_2$ is a neighborhood of $x$ then $N_1$ is also a neighborhood of $x$

③ If $N_1 \subseteq X$ & $N_2 \subseteq X$ are both neighborhoods of $x$ then so is $N_1 \cap N_2$ a neighborhood of $x$

④ If $N_1 \subseteq X$ is a neighborhood of $x$ then it contains neighborhood $N_2 \subseteq X$ of $x$ s.t $N_1$ is a neighborhood of each point of $N_2$

Definitions: In topological space, $\{x_i\}$ could converge to a limit $\lim_{i\to\infty} x_i$

$\lim_{i\to\infty} \frac{1}{i} = 0$

/en.wikipedia.org/wiki/Limit_point

Any neighborhood contains at least one other pt $x_k$ from $\{x_i\}$ eg $1/n$ for large $n$

$cl(S)$ when $S$ is a topological space

→ Should consist of $S$ union with

→ Should consist of $\lim_{i\to\infty} x_i$ for every convergent sequence $\{x_i\} \subseteq S$

For general topological space

$S$ is closed if $cl(S) = S$

$S$ is open if $S^c$ is closed

$\text{interior}(S) = \bigcup_{\substack{S'=\text{open} \\ S' \subseteq S}} S'$

$bnd(S) = cl(S) - int(S)$

$\stackrel{?}{=} cl(S) \cap cl(S^c)$

$relbnd(S) = cl(S) - relint(S)$

$relint(S) = \{x \mid x \in S \text{ s.t. } \exists \epsilon > 0 \text{ s.t. } \{y \mid \|x-y\| < \epsilon\} \cap aff(S) \subseteq S\}$

with norm $\|\cdot\|$

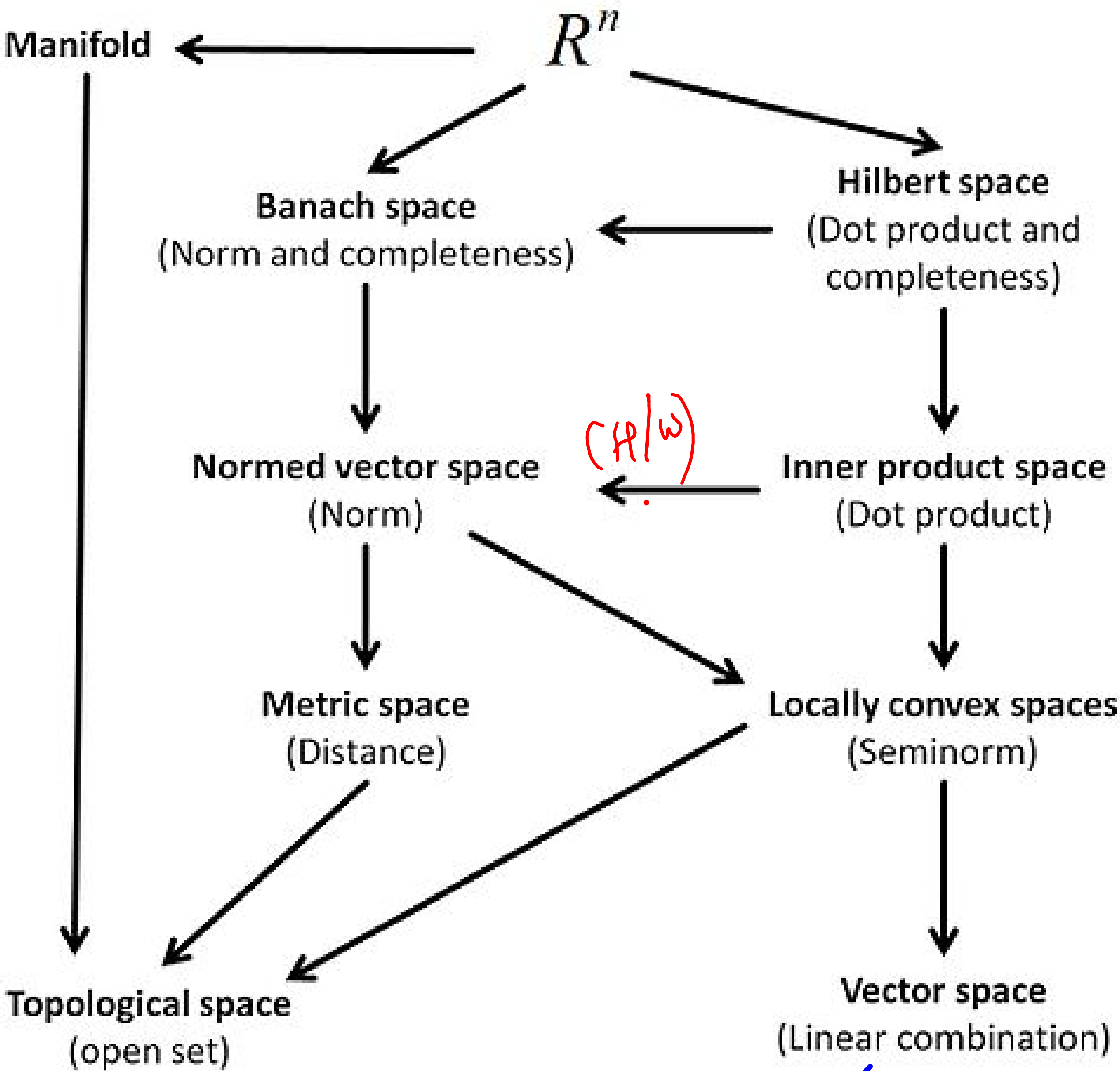
$\forall x \in S, \exists \epsilon > 0$ s.t. $\{y \mid \|y-x\| \leq \epsilon\} \subseteq S$ open set in Normed v.s

$bnd(S) = \partial(S)$

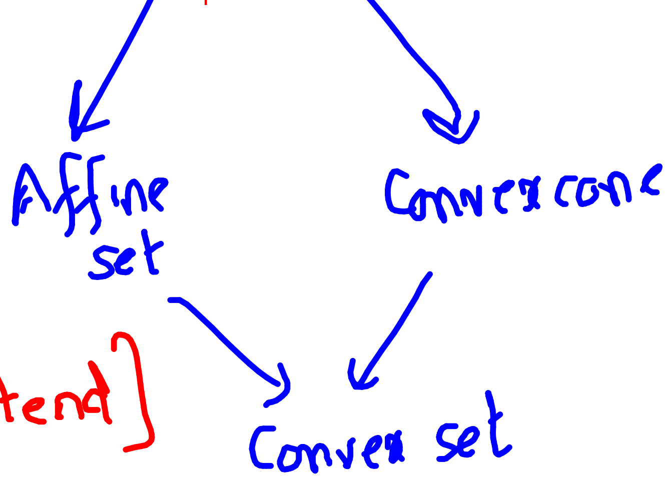

x belongs to the normed space

$cl(S) = \{x \mid \forall \epsilon > 0 \; S \cap \{y \mid \|x-y\| < \epsilon\} \neq \phi\}$

$int(S) = \{x \mid x \in S \text{ s.t. } \exists \epsilon > 0 \text{ s.t. } \{y \mid \|x-y\| < \epsilon\} \subseteq S\}$




Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.




[H/W: Confirm
or contend]

[H/w: Prove that "normed" space is a "metric" space]

Inner product space: It is a vector space over a field of scalars along with an inner product

eg: $\mathbb{R}$

an algebraic structure with addition, subtraction, multiplication & division

associative & commutative

must be commutative associative & distributive

multiplicative inverse must exist

ⓐ (Conjugate) symmetry:

$$\langle x,y\rangle = \langle y,x\rangle$$

ⓑ Linearity in the first argument

$$\langle ax,y\rangle = a\langle x,y\rangle$$

$$\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$$

ⓒ Positive definiteness:

$\langle x,x\rangle \geq 0$ with equality iff $x=0$

In general (see http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality)

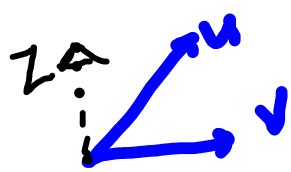
$$|\langle u,v\rangle| \leq \|u\|\,\|v\|$$ for any valid norm such as $\|\ \|_2$

Proof: If $v=0$, both sides are 0 & hence equality holds.

Assume $v \neq 0$ & let $z = u - \dfrac{\langle u,v\rangle}{\langle v,v\rangle} v$ {$z=0$ iff $u$ & $v$ are lin. dependent}

& $\therefore$ $\langle z,v\rangle = \left\langle u - \dfrac{\langle u,v\rangle}{\langle v,v\rangle} v, v\right\rangle = \langle u,v\rangle - \dfrac{\langle u,v\rangle}{\langle v,v\rangle}\langle v,v\rangle = 0$

(By linearity of the inner product in the first argument)




& $\therefore$ $\langle u,u\rangle = \|u\|^2 = \underbrace{\langle z,z\rangle + \left|\dfrac{\langle u,v\rangle}{\langle v,v\rangle}\right|^2 \langle v,v\rangle}_{\text{Substituting for } u = z + \frac{\langle u,v\rangle}{\langle v,v\rangle} v} + \underbrace{\left|\dfrac{\langle u,v\rangle}{\langle v,v\rangle}\right| \langle z,v\rangle}_{=0 \text{ from above}}$

$$= \|z\|^2 + \left(\frac{\langle u,v\rangle^2}{\|v\|^2}\right) \geq \frac{\langle u,v\rangle^2}{\|v\|^2}$$ {equality iff $z=0$}

$\Rightarrow$ $\boxed{\|u\|\,\|v\| \geq |\langle u,v\rangle|}$ → Cauchy Shwarz ineq.
{equality iff $u$ & $v$ are linearly dependent}

[H/w: Prove that "inner product space" is a "normed" vector space]

Inner product space: It is a vector space over a field of scalars along with an inner product

Assume $\mathbb{R}$ or complex

① $\langle x, x\rangle = \overline{\langle x, x\rangle} \Rightarrow \langle x, x\rangle$ must be real

$\therefore$ We can define $\|x\| = \sqrt{\langle x, x\rangle}$

We need to prove that $\|x\|$ is a valid norm

ⓐ • By defn of inner product, since $\langle x, x\rangle \geq 0$ with equality iff $x = 0$,

$\|x\| \geq 0$ iff $x = 0$

ⓑ • $\|tx\| = \sqrt{\langle tx, tx\rangle} = \sqrt{t \cdot \bar{t}\,\langle x, x\rangle}$

$= \sqrt{t \cdot \bar{t}}\,\|x\| = |t|\,\|x\|$ (For real & complex $t$, $|t| = \sqrt{t \times \bar{t}}$)

(c) $\|x+y\| = \sqrt{\langle x+y, x+y\rangle}$

$$= \sqrt{\langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle}$$

$$\leq \sqrt{\langle x,x\rangle + \langle y,y\rangle + \sqrt{\langle x,x\rangle\langle y,y\rangle} \times 2}$$

By cauchy Schwarz inequality

$$= \sqrt{(\|x\| + \|y\|)^2}$$

$$= \|x\| + \|y\|$$

– Hence proved that $\sqrt{\langle x,x\rangle}$ is a norm

$\Rightarrow$ Every inner product space is a normed space.

**Converse does not hold:** $\exists$ normed spaces that are not inner product spaces.

__Eg:__ $\|x\|_p = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{1/p}$