If $A > 0$ (positive definite) 1 & 3 dont assume Symmetry

1 $\mathrm{real}(\lambda) > 0$ Eg: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $x^TAx = x_1^2 + x_2^2$

2 If $A \in S^n$ (S^n is space of all symmetric matrices) then we can show that all its eigenvalues are real (H/W)

3 $x^TAx > 0 \ \forall x \in \mathbb{R}^n, x \neq 0$ & $x^TAx = 0$ iff $x = 0$

Assumes $A \in S^n$

4 x^TAy is an inner product (by virtue of defn of inner prod)

5 $A = LL^T$ L is lower triangular &

Assumes A is symmetric

$A = Q\Sigma Q^T$ where Q is orthonormal & Σ is positive diagonal matrix

6 $$A \doteq \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{anti-symmetric}}$$

$$x^TAx = \underbrace{\frac{1}{2}x^T(A + A^T)x}_{} + \underbrace{\frac{1}{2}x^T(A - A^T)x}_{0}$$

It does not hurt in convex analysis to consider only symmetric part of A ie to assume A is symmetric

$$x^TAx = (x^TAx)^T = x^TA^Tx$$

Positive definiteness from page 198
section 3.11 of http://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

$\|x\|_2 : \sqrt{\sum x_i^2}$

Euclidean balls and ellipsoids

(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

An ellipsoid is a Euclidean ball in a rotated space

ellipsoid: set of the form

As per this defn P being p.d is necessary

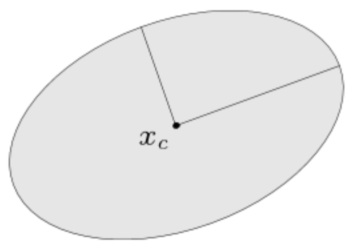
$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)

$P \succ 0$ if all its eigenvalues are > 0

$P = U \Sigma U^T$

$\left((x - x_c)^T U\right) \Sigma^{-1} \left((x - x_c)^T U\right)^T$



Write down relation between A & P

other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Scaling & rotation

Verify: $A = (U\Sigma^{1/2})$

Q: Is P being p.d necessary for convexity? For cone!

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \ge 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \le \|x\| + \|y\|$ (triangle inequality)

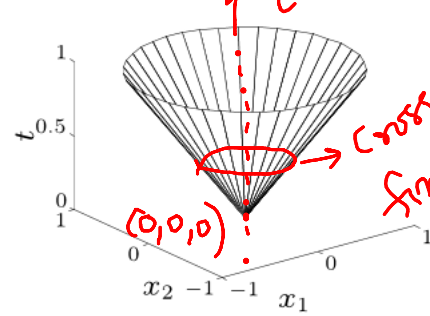
notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \le r\}$

pointed-ness

norm cone: $\{(x, t) \mid \|x\| \le t\}$

Euclidean norm cone is called second-order cone



t

(0,0,0)

Cross section fixed t

has cross section

Giving you a cross section as norm ball in $\mathbb{R}^2$

norm balls and cones are convex

$\|x\|_q \le \|x\|_p \le n^{1/p - 1/q} \|x\|_q \quad \forall \; 1 \le p \le q \le \infty$

Prove that under specific assumptions on P, $\sqrt{x^T P x}$ is a valid norm. Assume $x \in \mathbb{R}^n$ & $P \in \mathbb{R}^{n \times n}$

Proof: Suppose P is symmetric positive definite:

i.e $P^T = P$ & $\forall x \neq 0 \;\; x^T P x > 0$

The condition $\forall\, \mathbf{x} \neq \mathbf{0}, \sum_{i=1} \sum_{j=1} a_{ij}\mathbf{x}_i\mathbf{x}_j > 0$ involves a quadratic expression. The expression is guaranteed to be greater than $0 \;\forall\, \mathbf{x} \neq \mathbf{0}$ *iff* it can be expressed as $\sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{i-1} \beta_{ij} x_{ij} + x_{ii} \right)^2$, where $\lambda_i \geq 0$. This is possible *iff* A can be expressed as LDL^T, where, L is a lower traingular matrix with 1 in each diagonal entry and D is a diagonal matrix of all positive diagonal entries. Or equivalently, it should be possible to factorize A as RR^T, where $R = LD^{1/2}$ is a lower traingular matrix. Note that any symmetric matrix A can be expressed as LDL^T, where L is a lower traingular matrix with 1 in each diagonal entry and D is a diagonal matrix; positive definiteness has only an additional requirement that the diagonal entries of D be positive. This gives another equivalent condition for positive definiteness: *Matrix A is p.d. if and only if, A can be uniquely factored as $A = RR^T$, where R is a lower traingular matrix with positive diagonal entries.* This factorization of a p.d. matrix is reffered to as *Cholesky factorization.*

Source: pg 207 of http://www.cse.iitb.ac.in/~CS709/notes/LinearAlgebra.pdf

$$\Rightarrow x^T P x = x^T \underbrace{R R^T}_{\text{Assume } P = RR^T} x = \underbrace{(R^T x)^T}_{\text{Let } R^T x = y} (R^T x) = y^T y = \|y\|_2^2$$

$\therefore$ ① $x^T P x \geq 0$ since P is positive definite (By definition)

& $x^T P x = 0$ iff $x = 0$

② $\|\alpha x\|_P = \sqrt{(\alpha x)^T P (\alpha x)} = \sqrt{\alpha^2 x^T P x}$

$= |\alpha| \, \|x\|_P$

③ $\|x+y\|_P^2 = (x+y)^T P (x+y) = (x+y)^T R R^T (x+y)$

$= x^T R \underbrace{R^T x}_{u} + y^T R \underbrace{R^T y}_{v} + x^T R R^T y$

$+ y^T R R^T x$

$= u^T u + v^T v + u^T v + v^T u$

$= \|u\|_2^2 + \|v\|_2^2 + 2u^T v$

$(\|x\|_P + \|y\|_P)^2 = \underbrace{\|x\|_P}^2 + \underbrace{\|y\|_P} + 2\|x\|_P \|y\|_P$

$= \|u\|_2^2 + \|v\|_2^2 + 2\sqrt{\|u\|_2^2 \|v\|_2^2}$

$= \|u\|_2^2 + \|v\|_2 + 2\|u\|_2 \|v\|_2$

Rest follows from the Cauchy Schwarz inequality;

$2u^T v \leq \|u\|_2 \|v\|_2 \Rightarrow \|x+y\|_P \leq \|x\|_P + \|y\|_P$

Note a H/w problem for 3rd August:

Show that the following are vector spaces (assuming scalars come from a set S), and then answer questions that follow for each of them: Set of all matrices on S, set of all polynomials on S, set of all sequences of elements of S. (HINT: You can refer to this book for answers to most questions in this homework.) How would you understand the concepts of independence, span, basis, dimension and null space (chapter 2 of this book), eigenvalues and eigenvectors (chapter 5), inner product and orthogonality (chapter 6)? EXTRA: Now how about set of all random variables and set of all functions

Let us consider space of matrices:

$$\left\{ \begin{bmatrix} s_{11} \cdots s_{1m} \\ \\ s_{n1} \cdots s_{nm} \end{bmatrix} \middle| \; s_{11} \cdots s_{nm} \in S \right\}$$ over scalars S

So far we considered $S = \mathbb{R}$

Obvious that this is a vector space (since multiplication etc are defined on S)

For simplicity, let $S = \mathbb{R}$ & let us consider a norms for matrices, induced by norms for vectors

Let $N(x)$ be a vector norm satisfying the vector norm axioms:

Basis for vector space of matrices $(m \times n)$

$$\left\{ \underbrace{\begin{bmatrix} 1 & 0 & \cdot & 0 \\ \cdot & & & \\ \vdots & & & \\ 0 & \cdots & \cdot & 0 \end{bmatrix}}_{E_{11}} \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \cdots \\ & & & \\ 0 & & \cdot\ \cdot & 0 \end{bmatrix}}_{E_{12}} \cdots \underbrace{\begin{bmatrix} 0 & \cdot & \cdot & 0 \\ \cdot & & & \\ \vdots & & & \\ 0 & \cdot & \cdot & 1 \end{bmatrix}}_{E_{mn}} \right\}$$

$m \times n$ linearly independent elements that span the space of all matrices of size $m \times n$

$$B = \sum_{i,j} a_{ij} E_{ij} \equiv \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{mn} \end{bmatrix}$$

This vector $\in \mathbb{R}^{mn}$ / S^{mn} is a canonical representation of B

Then we will define a matrix norm

Can you prove that this is indeed a valid norm?

$\sup_{s} f(s) = \hat{f}$

$s \in S$, if $\hat{f}$ is minimum upper bnd

$$M_N(A) = \sup_{x \neq 0} \frac{N(Ax)}{N(x)}$$

as the matrix norm induced by vector norm $N(x)$

What, for example, will be

$M_N(I)$ → Ans: 1

irrespective of $N(x)$?

$$\left(\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right)$$

Examples

(a) If $N(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$

$\|A\|_1 = ?$

Ans: $\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}| |x_j|$

abs value of sum $\leq$ sum of abs values

Changing order of summation:

$$\|Ax\|_1 \leq \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$$

→ Let $C = \max_j \sum_{i=1}^{n} |a_{ij}|$

Then $\|Ax\|_1 \leq C\|x\|_1$

$$\Rightarrow \quad \|A\|_1 = \sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq C$$

But consider an $x = [0 \cdot 0 \cdot 1 \cdot \cdot 0\ 0]$

k^{th} position, where k is column index j for which $C = \sum_{i=1}^{n} |a_{ik}|$

Then $\|x\|_1 = 1$ & $\|Ax\|_1 = C$ (Show this)

$$\Rightarrow \quad \|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

i.e. if $N(x) = \|x\|_1$ then $M_N(A) = \max_j \sum_i |a_{ij}|$

(b) Similarly,

if $N(x) = \|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$

$$\|A\|_2 = [\text{dominant eigenvalue of } A^T A]^{1/2}$$

(c) If $N(x) = \|x\|_\infty = \max_i |x_i|$ $\left\{ \lim_{p\to\infty} \left(\sum |x_i|^p\right)^{1/p} \right.$

$$\|A\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$$

Other matrix norms:

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$$

Frobenius norm

Q: What abt inner products:

Note: Not all normed spaces are inner prod spaces.

Eg: $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$

→ for $p=2$ $\langle x,y\rangle = \sum_i x_i y_i$

→ For $p=1$ or ∞, No corresp. inner products

Read more on

http://www.math.ucsd.edu/~njw/Teaching/Math271C/Lecture_04.pdf

Eg of Frobenius inner product:

$$\langle A,B\rangle = \sum_i \sum_j a_{ij} b_{ij}$$

Weighted inner product

$$\langle A,B\rangle_W = \sum_i \sum_j a_{ij} b_{ij} w_{ij} \quad \text{for } w_{ij} > 0$$