Show that there does not exist $(x, y \in \mathbb{R}^n)$

$\langle x,y\rangle$ inner product s.t

$$\langle x,x\rangle = \left(\sum_{i=1}^{n} |x_i|^p\right)^{2/p}$$

V is a v.s

V is a normed vs

scalar mult

addition

0

$\|\cdot\|$

Scalar mult

addition

0

Cauchy sequence: (in any metric space)

A sequence is cauchy if its all its terms "eventually" become arbitrarily close to one another.

↳ ie given $\epsilon > 0$, $\exists$ N s.t if for any $m, n > N$ then $d(a_m, a_n) < \epsilon$

Q: Which of the following sequences are (a) Cauchy (b) Convergent?

(i) $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ in $\mathbb{R}$ (a) Cauchy & (b) Convergent

(ii) $\left(1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \ldots\right) = \left(\ldots \sum_{i=1}^{k}\left(\frac{1}{i}\right) \ldots\right)$ in $\mathbb{R}$

(b) $a_k \geq 1 + (\log_2 k)\ r(k) \xrightarrow[k \to \infty]{} \infty$ $\therefore$ Not convergent

(a) Not cauchy since $\mathbb{R}$ & not convergent?

(iii) $\left(1, \frac{1+\frac{2}{1}}{2}, \ldots x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2} \ldots\right)$ in $\mathbb{Q}$

(b) It is convergent in $\mathbb{R}$ (to $\sqrt{2}$) but NOT in $\mathbb{Q}$

(a) It is cauchy because it is convergent

Read: Babylonian method & Maclaurin series → for approx $\sin(x)$, $\cos(x)$

(iv) $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots\right)$ in $(0,2)$

(a) Cauchy by (i) (b) Not convergent in $(0,2)$

Idea: If x is an overestimate of $\sqrt{m}$

where m is a non-negative real no, then

$\frac{m}{x}$ is an underestimate of the square root:

ie if $x \geq \sqrt{m}$ then $\frac{m}{x} \leq \sqrt{m}$

$1.5 \geq \sqrt{2}$ $\quad$ $2/1.5 \leq \sqrt{2}$

$\Rightarrow \frac{1}{2}\left(x + \frac{m}{x}\right)$ will tend to approximate $\sqrt{m}$

Idea: $m = (x+e)^2 \Rightarrow e = \frac{m - x^2}{2x + e} \approx \frac{m - x^2}{2x}$ (if $e \ll x$)

$$\Rightarrow \boxed{x_{n+1} = x_n + e = \frac{x_n + \frac{m}{x_n}}{2}}$$

Claim: Any convergent sequence in a Metric space must be cauchy

Proof:

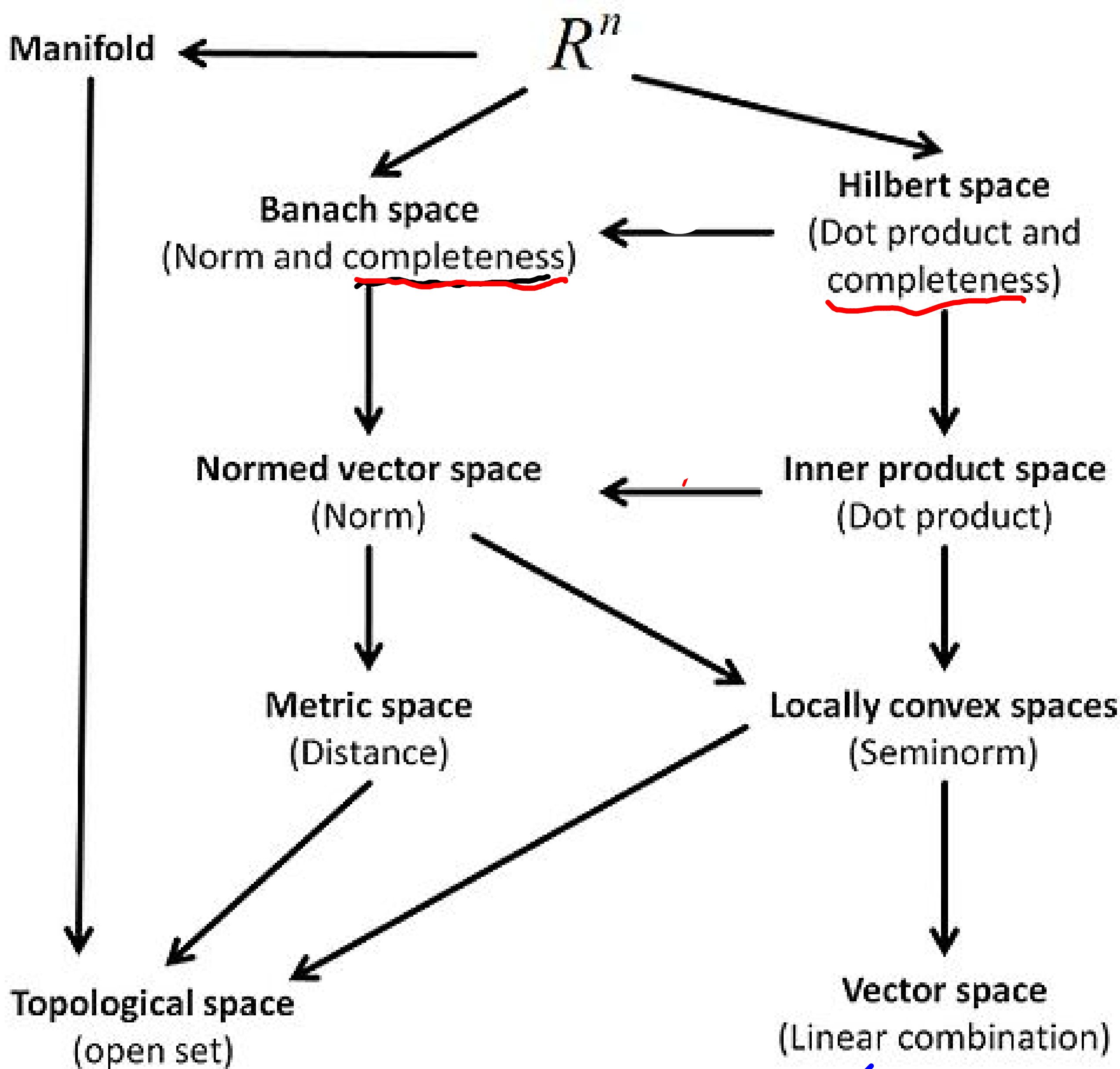
Let $(s_n) \to s$. Given $\epsilon > 0$ choose N s.t if $n > N$, we have $d(s_m, s_n) < \epsilon$

Then if $m, n > N$, $d(s_m, s_n) \leq d(s_m, s) + d(s, s_n)$

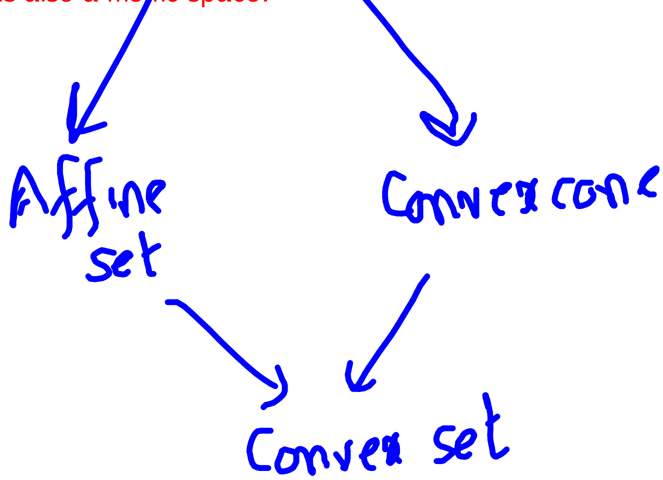
$< 2\epsilon$

BUT GIVEN A METRIC SPACE S, EVERY CAUCHY SEQUENCE NEED NOT CONVERGE TO A LIMIT POINT IN S!

(we saw several examples: $\left(x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2} \right)$,



Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



Complete metric space

A metric space S in which every cauchy sequence in S is convergent in S

Specialities of $\mathbb{R}^n$

① Every cauchy sequence is convergent

② A bounded sequence has atleast one limit point : Bolzano Weierstrass Theorem

Eg: $(1, 0, 1, 0, 1 \ldots)$

$x \in \mathbb{R}^n$ is said to be a limit point of $\{x_k\}$ if $\exists$ a subsequence of $\{x_k\}$ that converges to x.