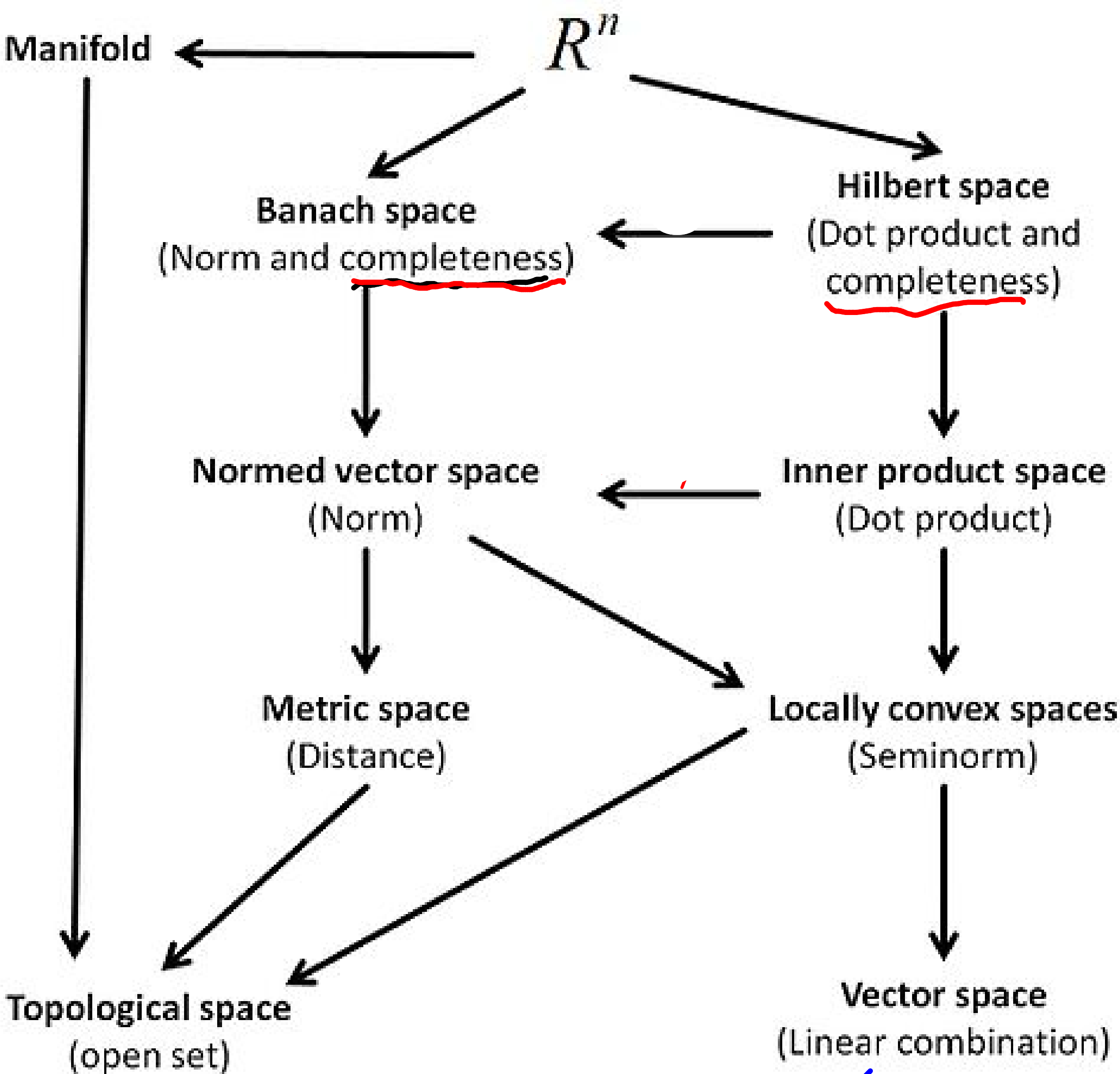
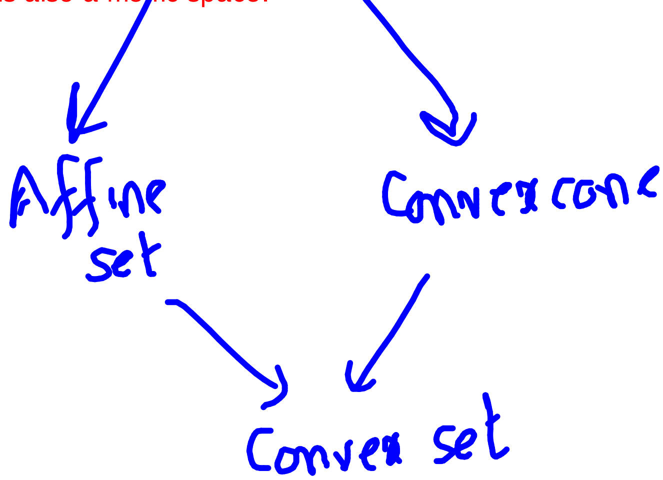


Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



Complete metric space

A metric space S in which every cauchy sequence in S is convergent in S

Other matrix norms:

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$$

Frobenius norm $= \sqrt{\mathrm{Trace}(A^T A)}$ → special case

① If A is symmetric: $\|A\|_F^2 = \lambda_1(A)^2 + \lambda_2(A)^2 + \dots + \lambda_n(A)^2$

② For general A: $\|A\|_F^2 = \sigma_1(A)^2 + \dots + \sigma_k(A)^2$

rank$(A) = k$.

(By virtue of trace

Q: What abt inner products:

Note: Not all normed spaces are inner prod spaces.

Eg: $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$ → for $p=2$: $\langle x, y\rangle = \sum_i x_i y_i$

→ For $p = 1$ or ∞, No corresp. inner products

Read more on

http://www.math.ucsd.edu/~njw/Teaching/Math271C/Lecture_04.pdf

Eg of Frobenius inner product:

$$\langle A, B\rangle = \sum_i \sum_j a_{ij} b_{ij}$$ → weighted inner product

$$\langle A, B\rangle_W = \sum_i \sum_j a_{ij} b_{ij} w_{ij} \quad \text{for } w_{ij} > 0$$

Hint: First prove (2)
You might use the orthonormal eigenvectors of A^TA as basis for column space of A and use this trick like in a previous lecture

Soln: Consider the j^{th} row a_j of A. Consider the singular value decomposition for A with $v_1\ v_2 \ldots v_k$ being orthonormal eigenvectors of A^TA

Then $\sum_{j=1}^{n} |a_j|^2 = \sum_{j=1}^{n} \sum_{r=1}^{d} a_{jr}^2$

$$= \sum_{j=1}^{n} \sum_{i=1}^{k} (a_j^T v_i)^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (a_j^T v_i)^2 = \sum_{i=1}^{k} |Av_i|^2 = \sum_{i=1}^{k} \sigma_i^2(A)$$

Singular values & Eigenvalues of A

$\underbrace{\text{Singular values}}_{\sigma}$ $\qquad$ $\underbrace{\text{Eigenvalues}}_{\lambda}$

$$Au = \sigma v$$
$$\underbrace{A^* v = \sigma u}$$
$$\Downarrow$$
$$A^* A u = \sigma^2 u$$

ie σ^2 is an eigenvalue of A^*A

$$Au = \lambda u$$

$$\boxed{\begin{array}{l} \text{if } A^T = A \text{ or } A^* = A \\ A^*Au = \lambda^2 u \\ \qquad\quad = \sigma^2 u \end{array}}$$

$l_p(S)$ = set of sequences with scalar field S that have finite p-norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} x_i^p\right)^{1/p}$$

$L_p(S)$ = set of measurable functions $f : X \to S$ with finite p-norm

$$\|f\|_p = \left(\int_{x \in X} f^p(x)\,dx\right)^{1/p}$$

Claim: $l_p(S)$ is complete (& so is $L_p(S)$)

Proof: Let $x_1 \ldots x_k \ldots$ be a Cauchy sequence in $l_p(S)$. s.t each

$x_n = (x_n^1, x_n^2 \ldots x_n^k \ldots)$

Claim: $x_n \ldots$ converges to x. Since $x_n \ldots$ is Cauchy, for any $\epsilon > 0$, $\exists\ N > 0$ s.t $\|x_n - x_m\|_{l_p} < \epsilon \quad \forall\ n, m > N$

We know (https://www.cse.iitb.ac.in/~cs709/notes/enotes/lecture5.pdf) that

$\|x\|_q \le \|x\|_p \le n^{1/p - 1/q} \|x\|_q \quad \forall\ 1 \le p \le q \le \infty$

$\Rightarrow$ if $q = \infty$, $\|x_n - x_m\|_{l_p} \ge \|x_n - x_m\|_\infty = \sup\limits_{k=1..\infty} |x_n^k - x_m^k|$

$\Rightarrow \forall\ \epsilon > 0, \exists\ N > 0$ s.t $|x_n^k - x_m^k| < \epsilon \quad \forall\ k$

Thus, $\forall k$, $x_1^k, x_2^k \ldots$ is a Cauchy sequence in $\mathbb{R}$

Since $\mathbb{R}$ is complete, $x_1^k, x_2^k \ldots$ must have a limit point x^k

<u>Claim</u> $(x^1, x^2 \ldots x^k \ldots)$ is the limit point of $x_1, x_2 \ldots$

This is easy to prove. Choose any $\epsilon > 0$. By replacing ϵ with $\epsilon/2$ in the previous discussion, we can find an $N > 0$ s.t $|x_n^k - x_m^k| < \epsilon/2 \quad \forall k \,\&\, \forall n, m > N$. Taking limits as $m \to \infty$, we get $|x_n^k - x_m| \leq \epsilon/2 \quad \forall k \,\&\, \forall n > N$

Taking supremum in k we get $\sup\limits_{k=1}^{\infty} |x_n^k - x_k| \leq \epsilon/2 \quad \forall n > N$

$\Rightarrow \|x_n - x\|_\infty \leq \epsilon/2 < \epsilon \quad \forall n > N \Rightarrow x_n$ converges to x

Choosing $\epsilon' = \epsilon(n^{1/p - 1/q})$ one can similarly get

$$\|x_n - x\|_p < \epsilon' \quad \forall n > N$$

Proof extended to completeness in L^p using measure theory at https://www.math.ucdavis.edu/~hunter/measure_theory/measure_notes_ch7.pdf

or at http://press.princeton.edu/chapters/s9627.pdf

$\ell_p(S)$ = set of sequences with scalar field S that have finite p-norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} x_i^p\right)^{1/p}$$

$L_p(S)$ = set of measurable functions $f: X \to S$ with finite p-norm

$$\|f\|_p = \left(\int_{x \in X} f^p(x)\, dx\right)^{1/p}$$

RECALL

A linear map/linear operator T between two vector spaces X & Y is $T: X \to Y$ s.t

$$T(\lambda x + \mu x') = \lambda Tx + \mu Tx'$$

$$\forall \lambda, \mu \in S$$
$$\forall x, x' \in X$$

If T is 1-1 & onto then T is called invertible. T^{-1} is defined s.t

$T^{-1} : Y \to X$ s.t $T^{-1} y = x$ iff $Tx = y$

ⓐ If X & Y are normed spaces

$$\|T\| = \sup_{\substack{x \neq 0 \\ x \in X}} \frac{\|Tx\|}{\|x\|}$$ is called operator norm

(Hint Keep relating to matrix norm)

T is called bounded if

$\exists N$ s.t $\|T\| \leq N \iff \exists M$ s.t $\|Tx\| \leq M\|x\|$ $\forall x \in X$

T is bounded iff it is continuous

Proof: Recall that T is called continuous if given any $\epsilon > 0$, $\exists$ a $\delta > 0$ s.t whenever

$\|x - x'\| \leq \delta$. $\|Tx - Tx'\| \leq \epsilon$ $x, x' \in X$

ⓐ Suppose $T : X \to Y$ is bounded. Then $\forall x, x' \in X$ we have

$$\|Tx - Tx'\| = \|T(x - x')\| \leq M\|x - x'\|$$

where M is s.t $\|Tx\| \le M\|x\|$ $\forall x \in X$

Taking $\delta = \epsilon/M$, we get that T is cts

(b) Suppose $T: X \to Y$ is continuous at all $x \in X$ including $0 \in X$. T being linear, we must have $T(0) = 0$. Let $\epsilon = 1$, then $\exists$ $\delta > 0$ s.t $\|Tx\| \le 1$ whenever $\|x\| \le \delta$.

For any $x \in X$ s.t $x \ne 0$, let $\tilde{x} = \delta \frac{x}{\|x\|}$

Now $\|\tilde{x}\| \le \delta \Rightarrow \|T\tilde{x}\| \le 1$. From linearity of T: $\|Tx\| = \frac{\|x\|}{\delta} \|T\tilde{x}\| \le M\|x\|$

where $M = \frac{1}{\delta}$. Thus, T is bounded

(b) If $X = \{f : D \to V\}$ is a space of functions from domain D to vector space V & $\boxed{T: V \to V}$ then f is called an eigenfunction of T & λ corresponding eigenvalue if

Like a square matrix

$$Tf = \lambda f$$

Relate to linear operators by relating them to (square) matrices

Note: Vector $v \in \mathbb{R}^n$ is a $fn: [1, \dots n] \to \mathbb{R}$ or $\mathbb{C}$, Seq s is a $fn: \mathbb{N} \to \mathbb{R}$

(c) If $T: X \to Y$ & T is 1-1 & onto then X and Y are said to be

LINEARLY ISOMORPHIC

If X & Y are Hilbert spaces then they are isomorphic if $\exists$ an orthogonal (unitary) linear map $U: X \to Y$.

That is U should satisfy

$$\langle x, x' \rangle_X = \langle Ux, Ux' \rangle_Y \quad \forall \; x, x' \in X$$

(d) If $T: X \to Y$

kernel of $T = \ker(T) = \{ x \in X \mid Tx = 0 \}$

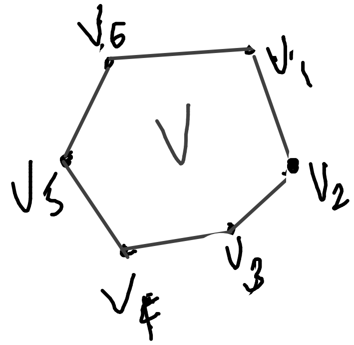
range of $T = \operatorname{ran}(T) = \{ y \in Y \mid \exists \; x \in X \text{ s.t. } Tx = y \}$

Both kernel and range are vector spaces

If X & Y are normed & T is bounded then $\ker(T)$ is closed

Intro to conic/convex duals through an eg:

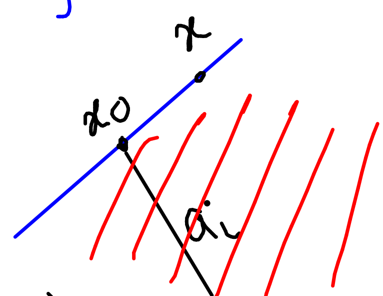
1) **V-polytope**: P is a V-polytope if $\exists$ a finite set of pts V s.t $P = \text{convex Hull}(V)$.



$V = \{v_1, v_2 .. v_6\}$ are vertices of the polytope P

2) **H-polytope**: P is an H-polytope if

$$P = \bigcap_{i=1}^{k} \{x \mid \langle a_i, x\rangle \geq b_i\}$$



$\langle x - x_0, a_i\rangle \geq 0$

s.t $\langle x_0, a_i\rangle = b_i$

One can see that every V-polytope is an H-polytope

But every H-polytope is NOT a V-polytope

However, if the H-polytope is closed, it is also a V-polytope

Both H & V polytopes are convex

Intersection of convex sets (half-spaces) is convex

Towards the generic "duality"

① Vector space $\rightarrow \left\{ x \mid x = \sum_i \alpha_i e_i, (e_1 \dots e_n) : \text{basis} \right\}$

$\rightarrow \left\{ x \mid \langle a_i, x \rangle = 0 \ \forall i \right\}$

② Affine space $\rightarrow \left\{ x \mid x = \sum_i \alpha_i e_i \ .. \ \sum \alpha_i = 1 \right\}$

$\rightarrow \left\{ x \mid \langle a_i, x \rangle = b_i \ \forall i \right\}$

③ closed polytopes $\rightarrow \left\{ x \mid x = \sum_i \alpha_i v_i \quad \sum \alpha_i = 1, \ \alpha_i \in [0,1] \right\}$

$\rightarrow \left\{ x \mid \langle a_i, x \rangle \geq b_i \ \forall i \right\}$

The parts in pink deal with characterization of the set in terms of linear operators $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ with $\langle a_i, x \rangle$ viewed as $A(x)$

(e) If X is normed v.s & Y is Banach then $T: X \to Y$ is Banach w.r.t the operator norm

(f) $T: X \to \mathbb{R}$ is called a linear functional

Then dual of X is set of all its linear functionals

Algebraic dual $= \{T \mid T: X \to \mathbb{R}\} = X^{\#}$

If X is finite dimensional, its dual $X^{\#}$ is linearly isomorphic to X

ie if $\{e_1 .. e_n\}$ is basis for X

then $\{g_i : X \to \mathbb{R}\}$ s.t

$$g_i\left(\sum_{j=1}^{n} x_j e_j\right) = x_i$$

form the basis for $X^{\#}$ so that

for any $g \in X^{\#}$, $g\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} g(e_i) x_i$

$\{g_1, g_2 .. g_n\}$ is called the dual basis

⑨ Topological dual $= \{T \mid T : X \to \mathbb{R}\} = X^*$

In finite dimensional case: $X^* = X^\#$ & X^* is isomorphic to X

T is a continuous linear functional

You get specific duals for subsets of vector spaces (such as convex sets, cones and affine sets) by putting restrictions on T.

Eg: If $C \subseteq X$ s.t X is a vector space

$$\boxed{\text{ⓐ } C^\# = \text{algebraic dual cone} = \{T \in X^\# \mid T(x) \geq 0 \ \forall \ x \in C\}}$$

in book represented as $\langle T, x \rangle$

ⓑ Further if X is a topological vector space & $C \subseteq X$

then

$C^* =$ topological dual cone

$= \{ T \in X^* \mid T(x) \geq 0 \ \forall x \in C \}$

Claims:

1. C^* is always a convex cone (irrespective of whether C is convex or cone or neither)

if $T_1 \in C^*$ & $T_2 \in C^*$ & $\theta_1, \theta_2 \geq 0$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \in X^*$$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \geq 0$$

$\Rightarrow C^*$ is a convex cone

(Similarly $C^\#$ is also always a convex cone)

(2) If X is finite dimensional,

$C^{\#} = C^*$

since $X^{\#} = X^*$

(3) If X is a Hilbert space,

C^* is closed .... More properties follow when X is a Hilbert space

Specialities of finite dimensional spaces

(i) Every finite dimensional normed vector space is a Banach space

(ii) Every linear operator on a finite dimensional vector space is bounded/continuous

. - . - H/W: Complete

(h) Riesz representation theorem:

As such we are looking at topological dual so T is cts/bnded

If $T: X \to \mathbb{R}$ and X is Hilbert and T is <u>bounded</u>, then $\exists$ a unique vector $y \in X$ s.t

$$T(x) = \langle y, x \rangle \quad \forall\, x \in X$$

$$X^* = \{ T_y(x) = \langle y, x \rangle \mid x \in X \}$$

is the dual of X^*

Defines a linear functional in terms of an inner product

Further, X & X^* are isomorphic.

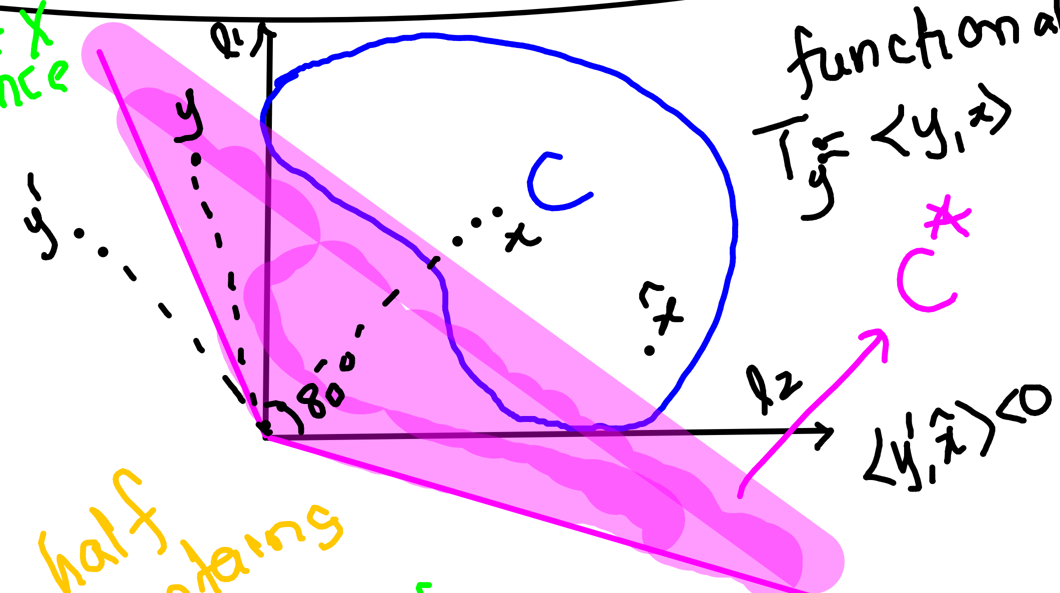
(i) Thus, if X is a Hilbert space over $\mathbb{R}$ as scalars and inner product $\langle \, , \rangle$, dual cone C^* of a set $C \subseteq X$ is

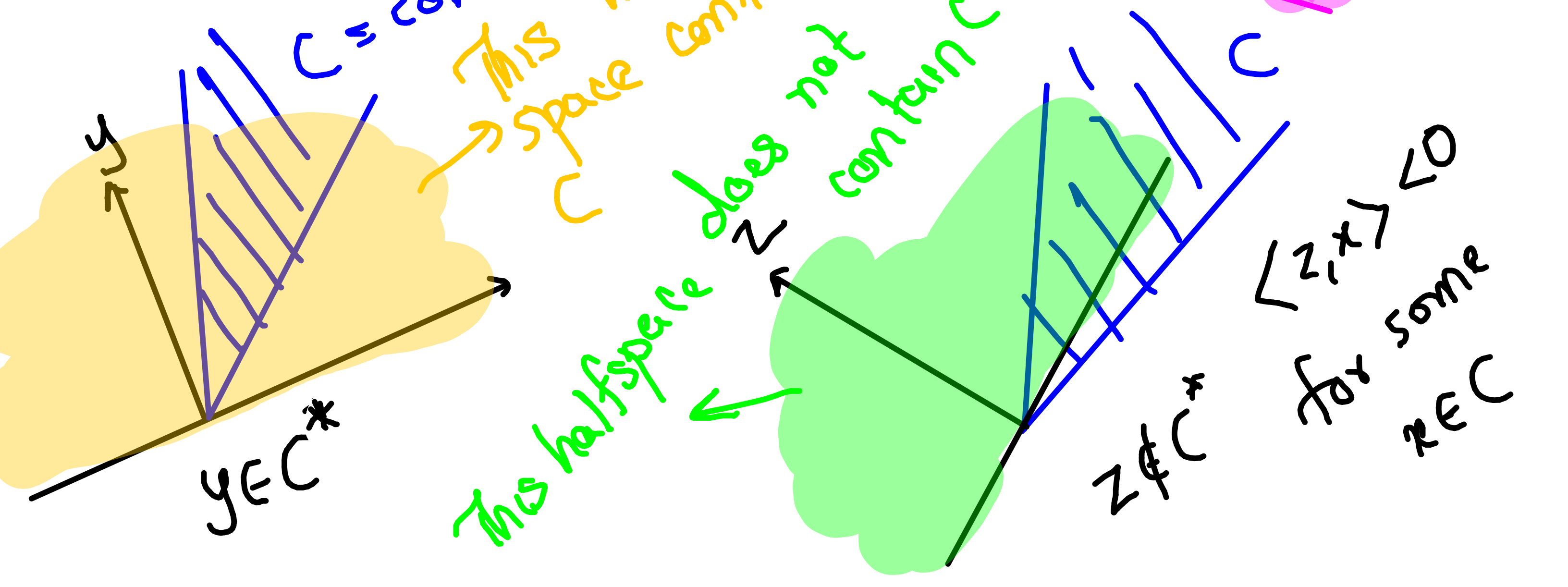
→ Since C^* is restricted to cts linear fnls, boundedness is guaranteed

$$C^* = \{ \underbrace{y \in X}_{T \in X^*} : \langle y, x \rangle \geq 0 \ \forall x \in C \}$$

since

In $\mathbb{R}^n$, etc this is intersection of half spaces





Properties of dual cones

① If X is a Hilbert space & $C \subseteq X$ then C^* is a closed convex cone

↳ We have already proved that C^* is a convex cone

↳ C^* = intersection of infinite closed topological half spaces

$$C^* = \bigcap_{x \in C} \{ y \mid y \in X, \langle y, x \rangle \geq 0 \}$$

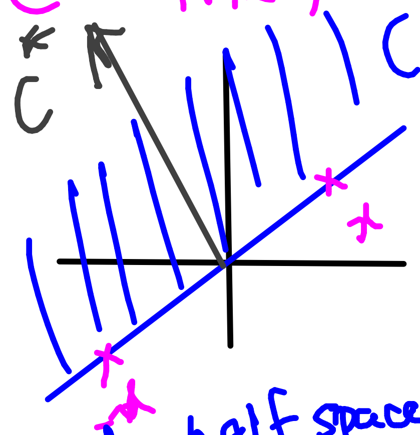
$\Rightarrow C^*$ is closed

② $C_1 \subseteq C_2 \Rightarrow C_2^* \subseteq C_1^*$

③ $\text{interior}(C^*) = \{ y \in X \mid \langle y, x \rangle > 0 \ \forall x \in X \}$

④ If C is a cone and has $\mathrm{int}(C) \neq \emptyset$ the C^* is pointed

↳ i.e. if $x \in C^*$ & $-x \in C^*$ then $x = 0$



⑤ If C is a cone then $\text{closure}(C) = C^{**}$

If C = open half space, C^{**} = closed half space

⑥ If closure of C is pointed then $\text{interior}(C^*) \neq \emptyset$

S is called conically spanning set of cone K iff $\text{conic}(S) = K$

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

Easy to prove it is a cone

$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

$\mathbf{S}^n_+$ is a convex cone

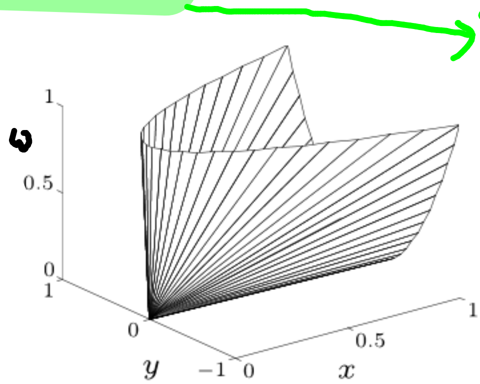
- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

Q: Is it convex? ✓

Is it a cone? ✗

Since $0 \notin S^n_{++}$

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$

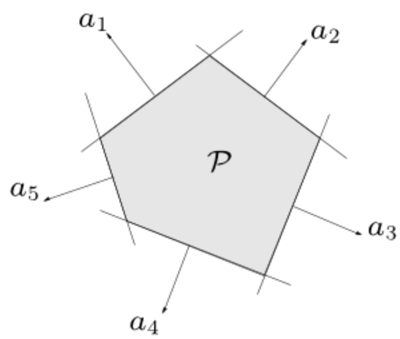


Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{p \times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

The Hahn Banach Thm: Any closed convex set in $\mathbb{R}^n$ is equivalent to intersection of all halfspaces that contain it

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}^n_+$ is a convex cone
- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$