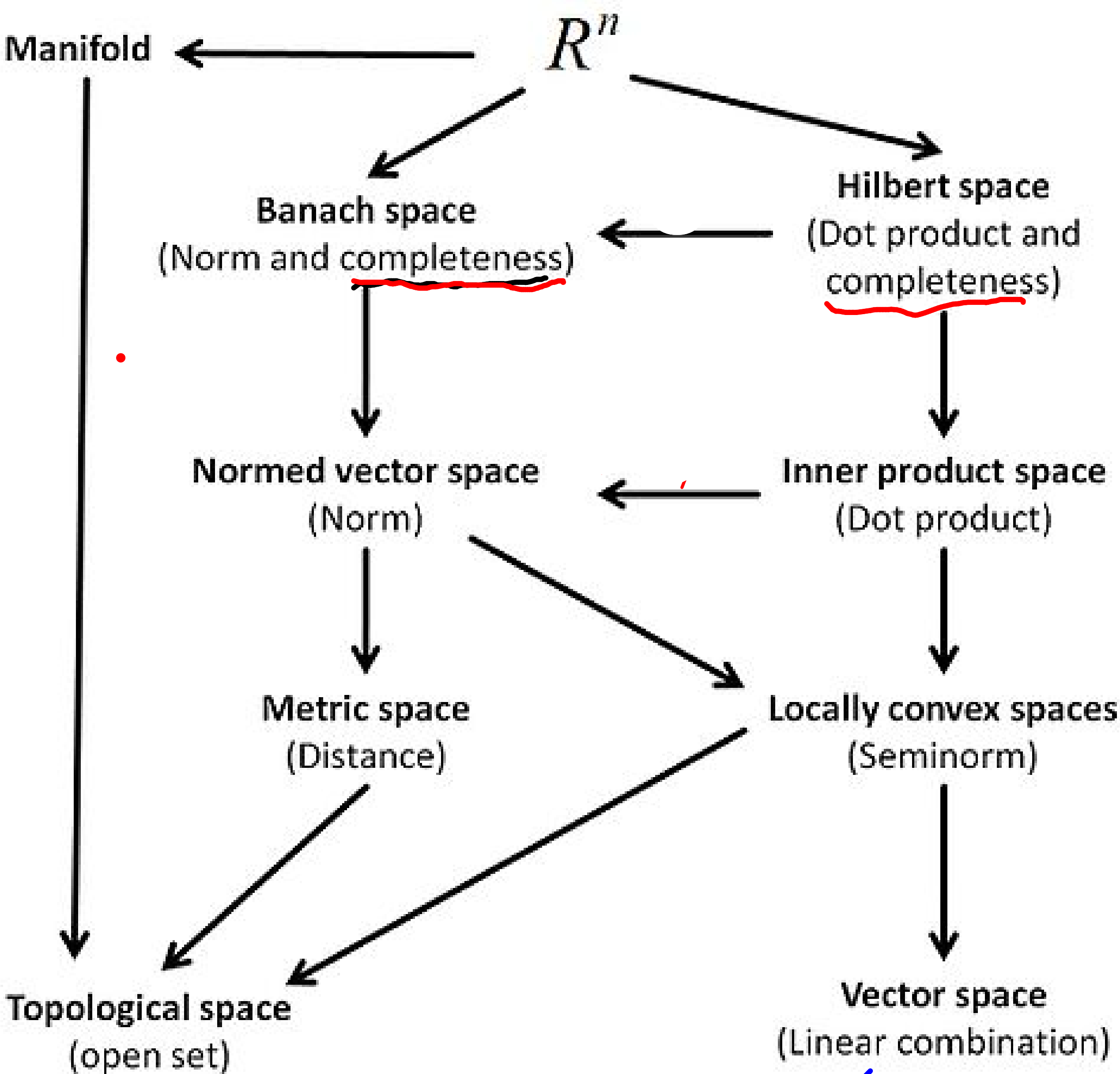
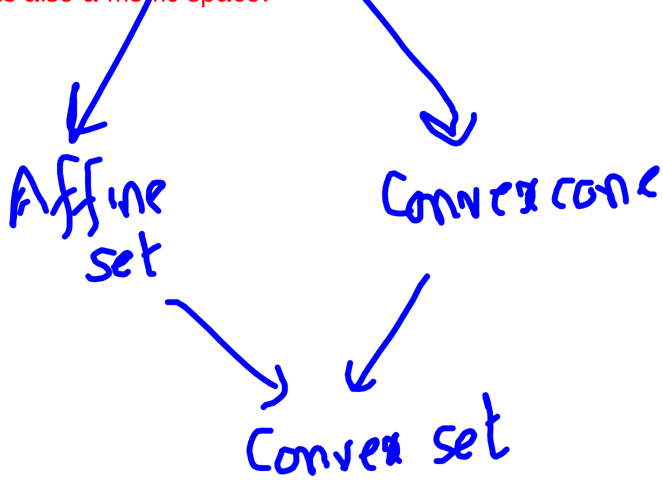


Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



Complete metric space

A metric space S in which every cauchy sequence in S is convergent in S

Towards the generic "duality"

① Vector space

$\to \{x \mid x = \sum_i \alpha_i e_i, (e_1 .. e_n) : \text{basis}\}$

$\to \{x \mid \langle a_i, x\rangle = 0 \ \forall i\}$

② Affine space

$\to \{x \mid x = \sum_i \alpha_i e_i \ .. \ \sum \alpha_i = 1\}$

$\to \{x \mid \langle a_i, x\rangle = b_i \ \forall i\}$

③ closed polytopes

$\to \{x \mid x = \sum_i \alpha_i v_i \quad \sum \alpha_i = 1, \ \alpha_i \in [0,1]\}$

$\to \{x \mid \langle a_i, x\rangle \geq b_i \ \forall i\}$

The parts in pink deal with characterization of the set in terms of linear operators $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ with $\langle a_i, x\rangle$ viewed as $A(x)$

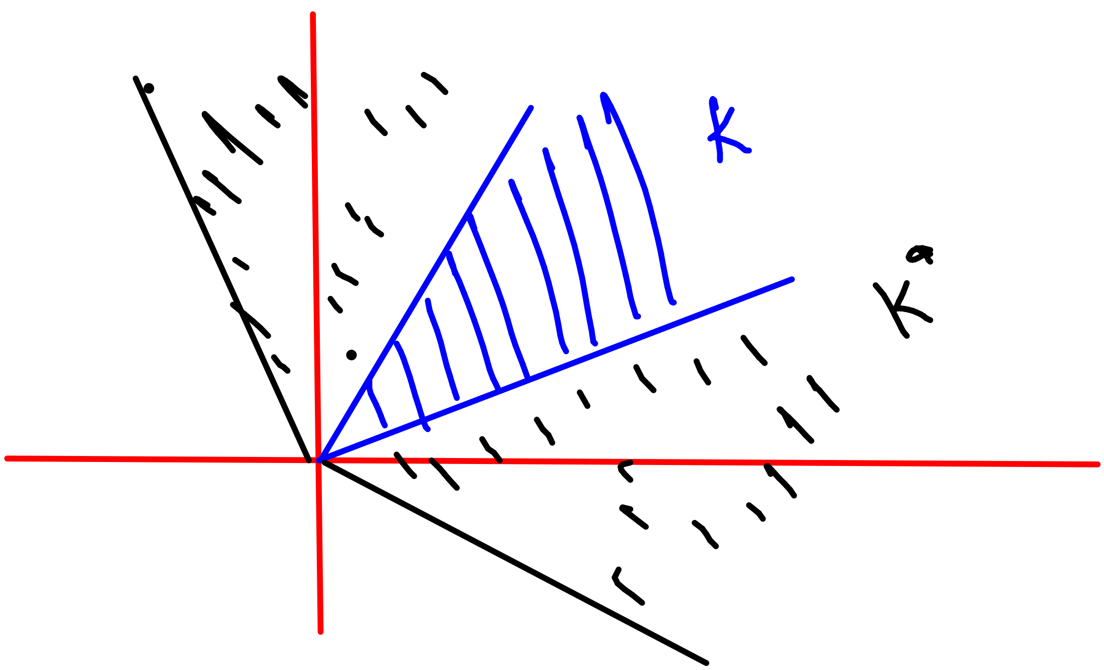
④ closed polyhedral cone

$\to \{x \mid x = \sum_i \alpha_i v_i \quad \alpha_i \geq 0\}$

$\to \{x \mid \langle a_i, x\rangle \geq 0 \ \forall i\}$

Motivated by this, we define. (for $C \subseteq X$ vector space)

Dual cone for set C: $C^* = \{a \mid \langle a, x\rangle \geq 0 \ \forall x \in C\}$



K^* is the dual cone of K. The boundary hyperplanes of K^* are orthogonal to the boundary hyperplanes of K

Q: What is dual cone of norm cone

$C = \{(x,t) \mid \|x\|_p \leq t\} \subseteq \mathbb{R}^{n+1}$

Soln: $C^* = \{a \mid \langle a, x\rangle \geq 0 \ \forall (x,t) \in C\}$

Claim:

$$C^* = \{(u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \le v\}$$

where $\|u\|_*$ = dual norm = $\sup\{u^T x \mid \|x\|_p \le 1\}$

operator norm

we show that

$$\langle x, u\rangle + tv \ge 0 \text{ for } \|x\|_p \le t \quad \text{A}$$

$$\Updownarrow$$

$$\|u\|_* \le v \quad \text{B}$$

B ⇒ A: Suppose $\|u\|_* \le v$ & $\|x\| \le t$ for some $t > 0$ (what happens if $t = 0$)

$\Rightarrow \langle u, -x/t\rangle \le \|u\|_* \le v$ ... ⇒ A

A ⇒ B: Let $\|u\|_* > v$ (ie by contradiction)

$\Rightarrow \exists$ an x with $\|x\| \le 1$ & $\langle x, u\rangle > v$

Taking $t = 1$, $\langle u, -x\rangle + v < 0$ which contradicts A

Further: If $p \in [1, \infty)$ then $\|u\|_* = \|u\|_q$ s.t $\frac{1}{p} + \frac{1}{q} = 1$ H/W

In particular, Euclidean norm is self dual: $\|u\|_* = \|u\|_2$

We can thus say the following:

① The boundary of the dual cone kind of helps characterize the dual description of the vector space/affine set/convex set/convex cone [verify]

② The polar of a set C is defined as

$$\text{polar of } C = C^{\circ} = \{a \mid \langle a, x\rangle \leq 0 \ \forall x \in C\} = -C^{*}$$

③ The polar of a convex set $C \subseteq \mathbb{R}^n$ can be interpreted as the projection into $\mathbb{R}^n$ of the polar $(C')^{\circ}$ of a cone $C' \subseteq \mathbb{R}^{n+1}$ s.t. C is the projection of C' into $\mathbb{R}^n$

$$\text{polar of convex set } C = C^{\circ} = \{a \mid \langle a, x\rangle \leq 1 \ \forall x \in C\}$$

Here, by projection, we mean cross-section in $\mathbb{R}^n$

(e) If X is normed v.s & Y is Banach then $T: X \to Y$ is Banach w.r.t the operator norm

(f) $T: X \to \mathbb{R}$ is called a linear functional

Then dual of X is set of all its linear functionals

Algebraic dual $= \{T \mid T: X \to \mathbb{R}\} = X^{\#}$

If X is finite dimensional, its dual $X^{\#}$ is linearly isomorphic to X

ie if $\{e_1 \ldots e_n\}$ is basis for X

then $\{g_i : X \to \mathbb{R}\}$ s.t

$$g_i\left(\sum_{j=1}^{n} x_j e_j\right) = x_i$$

form the basis for $X^{\#}$ so that

for any $g \in X^{\#}$, $g\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} g(e_i) x_i$

$\{g_1, g_2 \ldots g_n\}$ is called the dual basis

(9) Topological dual $= \{T \mid T : X \to \mathbb{R}\} = X^*$

In finite dimensional case: $X^* = X^\#$ & X^* is isomorphic to X

T is a continuous linear functional

You get specific duals for subsets of vector spaces (such as convex sets, cones and affine sets) by putting restrictions on T.

Eg: If $C \subseteq X$ s.t X is a vector space

(a) $C^\#$ = algebraic dual cone $= \{T \in X^\# \mid T(x) \geq 0 \;\forall\; x \in C\}$ in book represented as $\langle T, x \rangle$

(b) Further if X is a topological vector space & $C \subseteq X$

then

$C^* =$ topological dual cone

$$= \{ T \in X^* \mid T(x) \geq 0 \ \forall x \in C \}$$

Claims:

① C^* is always a convex cone

(irrespective of whether C is convex or cone or neither)

if $T_1 \in C^*$ & $T_2 \in C^*$ & $\theta_1, \theta_2 \geq 0$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \in X^*$$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \geq 0$$

$\Rightarrow C^*$ is a convex cone

(Similarly $C^\#$ is also always a convex cone)

② If X is finite dimensional,

$$C^{\#} = C^{*}$$

since $X^{\#} = X^{*}$

③ If X is a Hilbert space,

C^{*} is closed .... More properties follow when X is a Hilbert space

Specialities of finite dimensional spaces

(i) Every finite dimensional normed vector space is a Banach space

(ii) Every linear operator on a finite dimensional vector space is bounded/continuous

. - . - H/W: Complete