We have already seen that the class of functions f for fixed domain D & range vector space V themselves form a vector space

$$f : D \to V$$

Next we see another class of functions (convex functions) which is convex cone

3. Convex functions

- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions
- convexity with respect to generalized inequalities

Definition

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

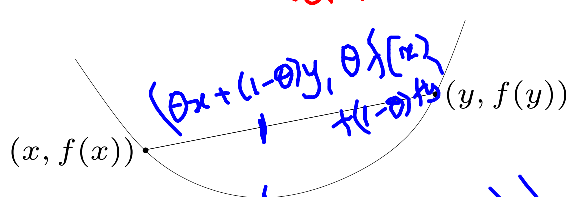
→ so that $\theta x + (1-\theta)y \in \mathbf{dom}\, f$ $\forall\, x, y \in \mathbf{dom}\, f$

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

fn at convex combination

convex combination of fns at x & y

for all $x, y \in \mathbf{dom}\, f$, $0 \le \theta \le 1$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \ne y$, $0 < \theta < 1$

Epigraph and sublevel set

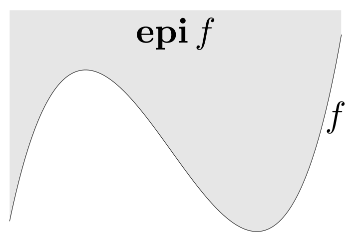
α-**sublevel set** of $f : \mathbf{R}^n \to \mathbf{R}$:

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

epigraph of $f : \mathbf{R}^n \to \mathbf{R}$:

$$\mathbf{epi}\, f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\}$$



f is convex if and only if $\mathbf{epi}\, f$ is a convex set

The family of convex functions is a convex cone } Q1

3–11

If f is convex, is C_α convex?
If f is convex, is epi$f$ convex? } Q2

Does convexity of C_α or epi$f$ imply convexity of f? } Q3

Examples on R

convex:

- affine: $ax + b$ on **R**, for any $a, b \in$ **R** ✓
- exponential: e^{ax}, for any $a \in$ **R** AM $\geq$ GM
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on **R**, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

concave:

- affine: $ax + b$ on **R**, for any $a, b \in$ **R**
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m \times n}$

affine functions are convex and concave; all norms are convex

examples on $\mathbf{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

examples on $\mathbf{R}^{m \times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

$$\max_v \frac{\|Xv\|_2}{\|v\|_2}$$