Definition 22 **[Directional derivative]:** *The directional derivative of $f(\mathbf{x})$ at $\mathbf{x}$ in the direction of the unit vector $\mathbf{v}$ is*

$$D_{\mathbf{v}}f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x}+h\mathbf{v}) - f(\mathbf{x})}{h} \tag{4.12}$$

provided the limit exists.

PAGES 231 TO 239 OF http://www.cse.iitb.ac.in/~ cs709/notes/BasicsOfConvexOptimization.pdf

As a special case, when $\mathbf{v} = \mathbf{u}^k$ the directional derivative reduces to the partial derivative of f with respect to x_k.

$$D_{\mathbf{u}^k}f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_k}$$

Theorem 57 *If $f(\mathbf{x})$ is a differentiable function of $\mathbf{x} \in \Re^n$, then f has a directional derivative in the direction of any unit vector $\mathbf{v}$, and*

$$D_{\mathbf{v}}f(\mathbf{x}) = \sum_{k=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_k} v_k \tag{4.13}$$

Definition 23 **[Gradient Vector]:** *If f is differentiable function of $\mathbf{x} \in \Re^n$, then the gradient of $f(\mathbf{x})$ is the vector function $\nabla f(\mathbf{x})$, defined as:*

$$\nabla f(\mathbf{x}) = [f_{x_1}(\mathbf{x}),\ f_{x_2}(\mathbf{x}), \ldots,\ f_{x_n}(\mathbf{x})]$$

The directional derivative of a function f at a point $\mathbf{x}$ in the direction of a unit vector $\mathbf{v}$ can be now written as

Theorem 58 *Suppose f is a differentiable function of $\mathbf{x} \in \Re^n$. The maximum value of the directional derivative $D_{\mathbf{v}} f(\mathbf{x})$ is $||\nabla f(\mathbf{x}||$ and it is so when $\mathbf{v}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.*

What does the gradient $\nabla f(\mathbf{x})$ tell you about the function $f(\mathbf{x})$? We will illustrate with some examples. Consider the polynomial $f(x,y,z) = x^2y + z\sin xy$ and the unit vector $\mathbf{v}^T = \frac{1}{\sqrt{3}}[1,1,1]^T$. Consider the point $p_0 = (0,1,3)$. We will compute the directional derivative of f at p_0 in the direction of $\mathbf{v}$. To do this, we first compute the gradient of f in general: $\nabla f = \big[2xy + yz\cos xy,\ x^2 + xz\cos xy,\ \sin xy$ Evaluating the gradient at a specific point p_0, $\nabla f(0,1,3) = [3,\ 0,\ 0]^T$. The directional derivative at p_0 in the direction $\mathbf{v}$ is $D_{\mathbf{v}} f(0,1,3) = [3,0,0].\frac{1}{\sqrt{3}}[1,1,1]^T = \sqrt{3}$. This directional derivative is the rate of change of f at p_0 in the direction $\mathbf{v}$; it is positive indicating that the function f increases at p_0 in the direction $\mathbf{v}$. All our ideas about first and second derivative in the case of a single variable carry over to the directional derivative.

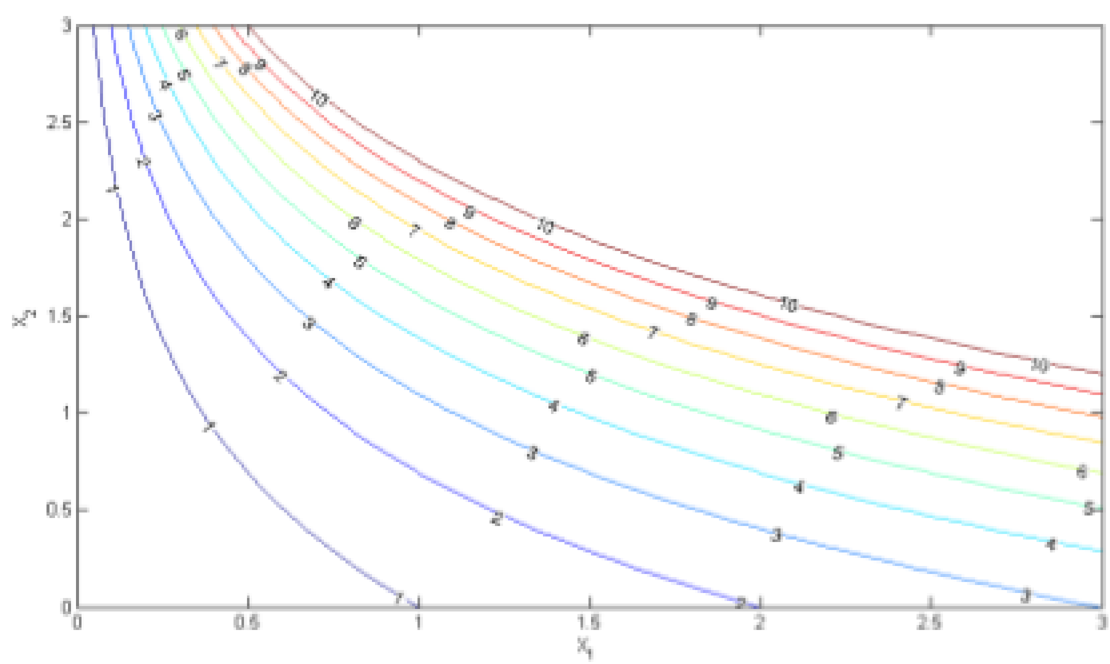

Figure 4.12: 10 level curves for the function $f(x_1,x_2) = x_1 e^{x_2}$.

Consider the function $f(x_1,x_2) = x_1 e^{x_2}$. Figure 4.12 shows 10 level curves for this function, corresponding to $f(x_1,x_2) = c$ for $c = 1,2,\ldots,10$. The idea behind a level curve is that as you change $\mathbf{x}$ along any level curve, the function value remains unchanged, but as you move $\mathbf{x}$ across level curves, the function value changes.

Theorem 59 *Let $f : \mathcal{D} \rightarrow \Re$ with $\mathcal{D} \in \Re^n$ be a differentiable function. The gradient ∇f evaluated at $\mathbf{x}^*$ is orthogonal to the tangent hyperplane (tangent line in case $n = 2$) to the level surface of f passing through $\mathbf{x}^*$.*

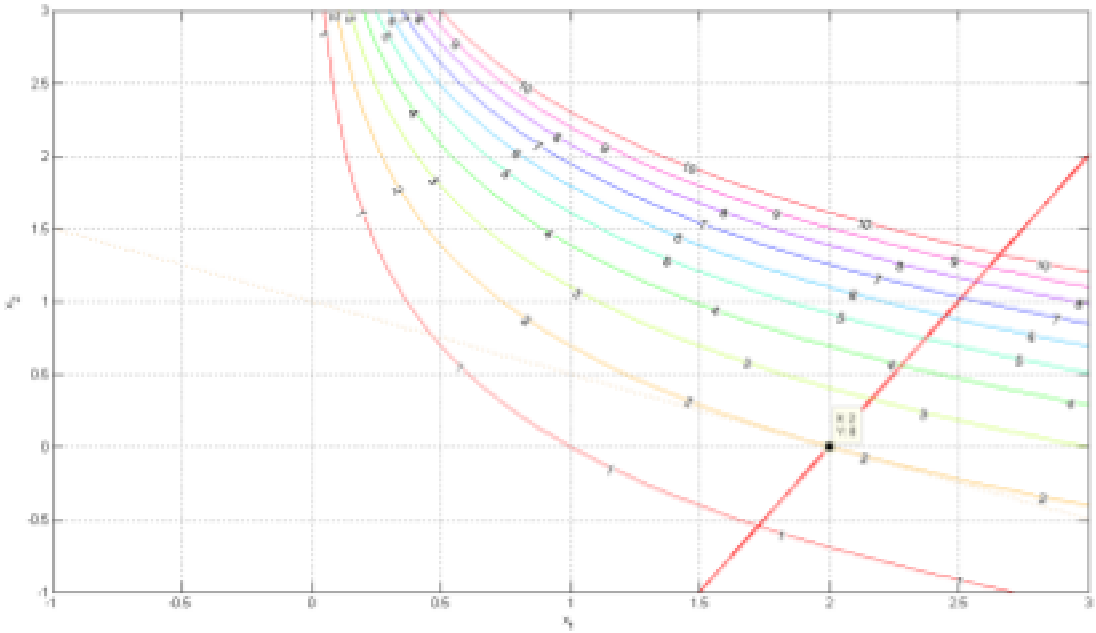

Figure 4.13: The level curves from Figure 4.12 along with the gradient vector at $(2, 0)$. Note that the gradient vector is perpenducular to the level curve $x_1 e^{x_2} = 2$ at $(2, 0)$.

Consider the same plot as in Figure 4.12 with a gradient vector at $(2, 0)$ as shown in Figure 4.13. The gradient vector $[1,\ 2]^T$ is perpendicular to the tangent hyperplane to the level curve $x_1 e^{x_2} = 2$ at $(2, 0)$. The equation of the tangent hyperplane is $(x_1 - 2) + 2(x_2 - 0) = 0$ and it turns out to be a tangent line.

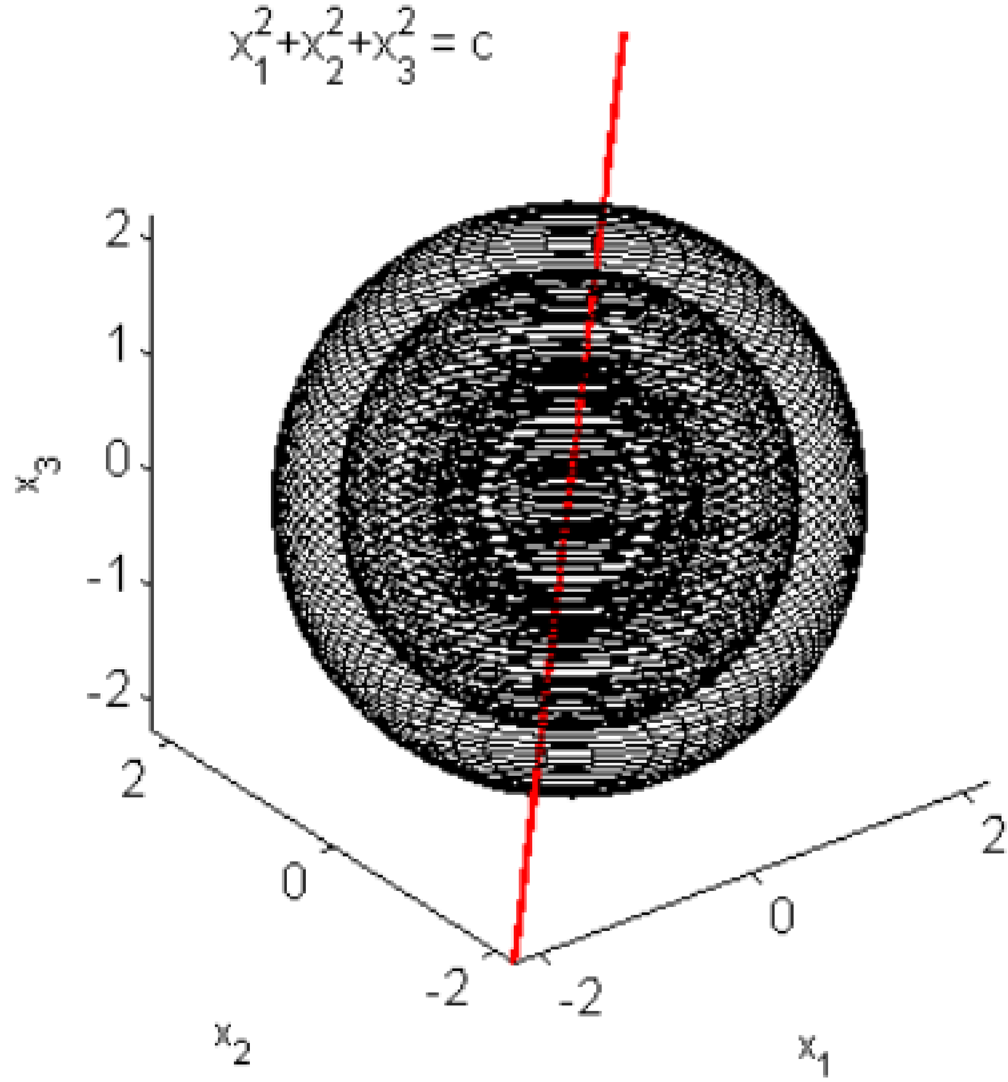

Figure 4.14: 3 level surfaces for the function $f(x_1, x_2, x_3) = x_1^2+x_2^2+x_3^2$ with $c = 1, 3, 5$. The gradient at $(1, 1, 1)$ is orthogonal to the level surface $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = 3$ at $(1, 1, 1)$.

The level surfaces for $f(x_1, x_2, x_3) = x_1^2+x_2^2+x_3^2$ are shown in Figure 4.14. The gradient at $(1, 1, 1)$ is orthogonal to the tangent hyperplane to the level surface $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = 3$ at $(1, 1, 1)$. The gradient vector at $(1, 1, 1)$ is $[2,\ 2,\ 2]^T$ and the tanget hyperplane has the equation $2(x_1 - 1) + 2(x_2 - 1) + 2(x_3 - 1) = 0$, which is a plane in $3D$. On the other hand, the dotted line in Figure 4.15 is not orthogonal to the level surface, since it does not coincide with the gradient.

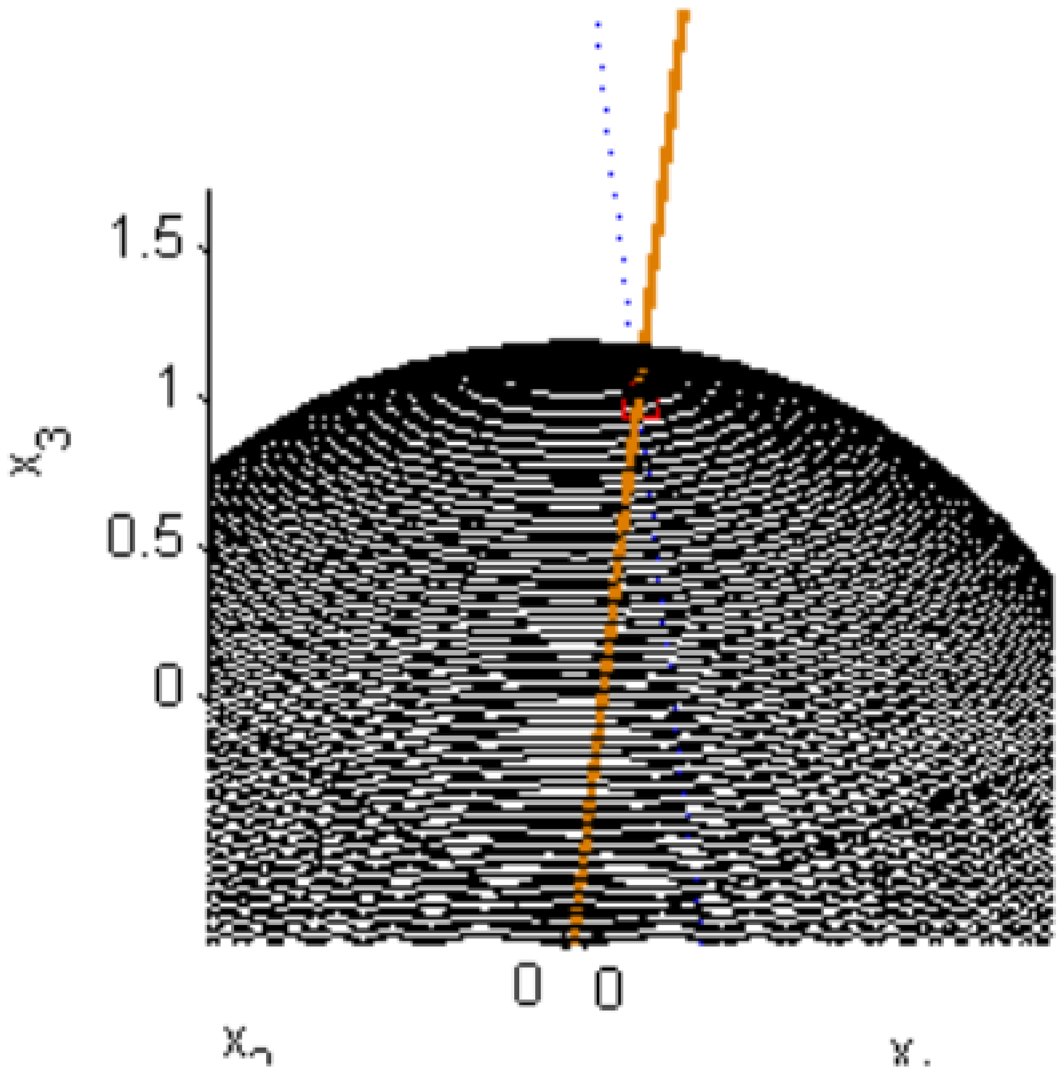

Figure 4.15: Level surface $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = 3$. The gradient at $(1, 1, 1)$, drawn as a bold line, is perpendicular to the tangent plane to the level surface at $(1, 1, 1)$, whereas, the dotted line, though passing through $(1, 1, 1)$ is not perpendicular to the same tangent plane.

3. Let $f(x_1, x, x_3) = x_1^2 x_2^3 x_3^4$ and consider the point $\mathbf{x}^0 = (1, 2, 1)$. We will find the equation of the tangent plane to the level surface through $\mathbf{x}^0$. The level surface through $\mathbf{x}^0$ is determined by setting f equal to its value evaluated at $\mathbf{x}^0$; that is, the level surface will have the equation $x_1^2 x_2^3 x_3^4 = 1^2 2^3 1^4 = 8$. The gradient vector (normal to tangent plane) at

   $(1, 2, 1)$ is $\nabla f(x_1, x_2, x_3)|_{(1,2,1)} = [2x_1 x_2^3 x_3^4,\ 3x_1^2 x_2^2 x_3^4, 4x_1^2 x_2^3 x_3^3]^T\Big|_{(1,2,1)} = [16,\ 12,\ 32]^T$. The equation of the tangent plane at $\mathbf{x}^0$, given the normal vector $\nabla f(\mathbf{x}^0)$ can be easily written down: $\nabla f(\mathbf{x}^0)^T.[\mathbf{x} - \mathbf{x}^0] = 0$ which turns out to be $16(x_1 - 1) + 12(x_2 - 2) + 32(x_3 - 1) = 0$, a plane in $3D$.

4. Consider the function $f(x, y, z) = \frac{x}{y+z}$. The directional derivative of f in the direction of the vector $\mathbf{v} = \frac{1}{\sqrt{14}}[1,\ 2,\ 3]$ at the point $x^0 = (4, 1, 1)$ is $\nabla^T f\Big|_{(4,1,1)} \cdot \frac{1}{\sqrt{14}}[1,\ 2,\ 3]^T = \left[\frac{1}{y+z},\ -\frac{x}{(y+z)^2},\ -\frac{x}{(y+z)^2}\right]\Big|_{(4,1,1)} \cdot \frac{1}{\sqrt{14}}[1,\ 2,\ 3]^T = \left[\frac{1}{2},\ -1,\ -1\right] \cdot \frac{1}{\sqrt{14}}[1,\ 2,\ 3]^T = -\frac{9}{2\sqrt{14}}$. The directional derivative is negative, indicating that the function decreases along the direction of $\mathbf{v}$. Based on theorem 58, we know that the maximum rate of change of a function at a point $\mathbf{x}$ is given by $||\nabla f(\mathbf{x})||$ and it is in the direction $\frac{\nabla f(\mathbf{x})}{||\nabla f(\mathbf{x})||}$. In the example under consideration, this maximum rate of change at $\mathbf{x}^0$ is $\frac{3}{2}$ and it is in the direction of the vector $\frac{2}{3}\left[\frac{1}{2},\ -1,\ -1\right]$.

5. Let us find the maximum rate of change of the function $f(x, y, z) = x^2 y^3 z^4$ at the point $\mathbf{x}^0 = (1, 1, 1)$ and the direction in which it occurs. The gradient at $\mathbf{x}^0$ is $\nabla^T f\Big|_{(1,1,1)} = [2,\ 3,\ 4]$. The maximum rate of change at $\mathbf{x}^0$ is therefore $\sqrt{29}$ and the direction of the corresponding rate of change is $\frac{1}{\sqrt{29}}[2,\ 3,\ 4]$. The minimum rate of change is $-\sqrt{29}$ and the corresponding direction is $-\frac{1}{\sqrt{29}}[2,\ 3,\ 4]$.

6. Let us determine the equations of (a) the tangent plane to the paraboloid $\mathcal{P} : x_1 = x_2^2 + x_3^2 + 2$ at $(-1, 1, 0)$ and (b) the normal line to the tangent plane. To realize this as the level surface of a function of three variables, we define the function $f(x_1, x_2, x_3) = x_1 - x_2^2 - x_3^2$ and find that the paraboloid $\mathcal{P}$ is the same as the level surface $f(x_1, x_2, x_3) = -2$. The normal to the tangent plane to $\mathcal{P}$ at $\mathbf{x}^0$ is in the direction of the gradient vector $\nabla f(\mathbf{x}^0) = [1, -2, 0]^T$ and its parametric equation is $[x_1,\ x_2,\ x_3] = [-1+t,\ 1-2t,\ 0]$. The equation of the tangent plane is therefore $(x_1 + 1) - 2(x_2 - 1) = 0$.

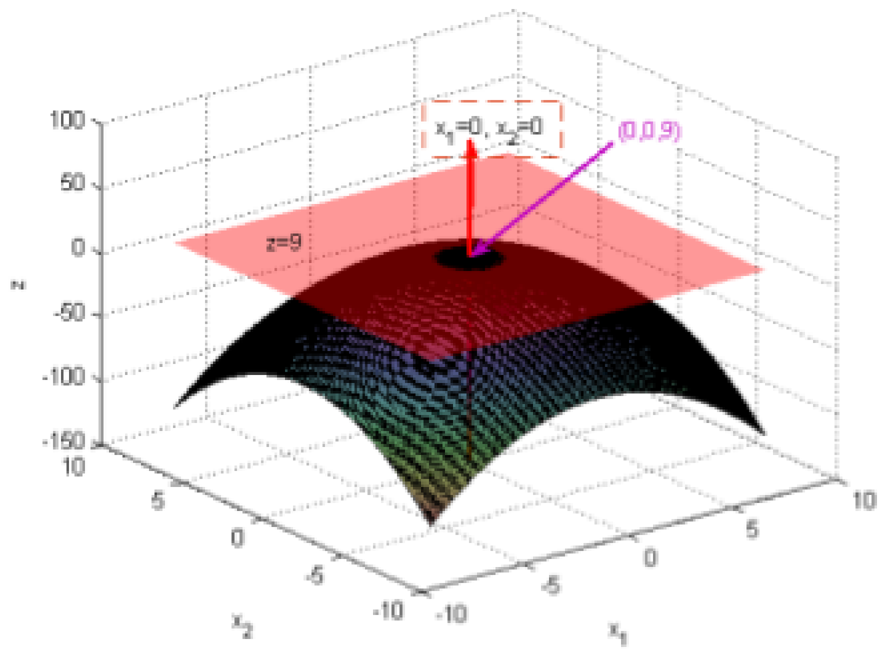

Figure 4.17: The paraboloid $f(x_1, x_2) = 9 - x_1^2 - x_2^2$ attains its maximum at $(0, 0)$. The tanget plane to the surface at $(0, 0, f(0, 0))$ is also shown, and so is the gradient vector ∇F at $(0, 0, f(0, 0))$.

We can embed the graph of a function of n variables as the 0-level surface of a function of $n + 1$ variables. More concretely, if $f : \mathcal{D} \to \Re$, $\mathcal{D} \subseteq \Re^n$ then we define $F : \mathcal{D}' \to \Re$, $\mathcal{D}' = \mathcal{D} \times \Re$ as $F(\mathbf{x}, z) = f(\mathbf{x}) - z$ with $\mathbf{x} \in \mathcal{D}'$. The function f then corresponds to a single level surface of F given by $F(\mathbf{x}, z) = 0$. In other words, the $0-$level surface of F gives back the graph of f. The gradient of F at any point $(\mathbf{x}, z)$ is simply, $\nabla F(\mathbf{x}, z) = [f_{x_1}, f_{x_2}, \ldots, f_{x_n}, -1]$ with the first n components of $\nabla F(\mathbf{x}, z)$ given by the n components of $\nabla f(\mathbf{x})$. We note that the level surface of F passing through point $(\mathbf{x}^0, f(\mathbf{x}^0)$ is its 0-level surface, which is essentially the surface of the function $f(\mathbf{x})$. The equation of the tangent hyperplane to the $0-$level surface of F at the point $(\mathbf{x}^0, f(\mathbf{x}^0)$ (that is, the tangent hyperplane to $f(\mathbf{x})$ at the point $\mathbf{x}_0$), is $\nabla F(\mathbf{x}^0, f(\mathbf{x}^0))^T.[\mathbf{x} - \mathbf{x}^0, z - f(\mathbf{x}^0)]^T = 0$. Substituting appropriate expression for $\nabla F(\mathbf{x}^0)$, the equation of the tangent plane can be written as

$$\left(\sum_{i=1}^{n} f_{x_i}(\mathbf{x}^0)(x_i - x_i^0)\right) - (z - f(\mathbf{x}^0)) = 0$$

or equivalently as,

$$\left(\sum_{i=1}^{n} f_{x_i}(\mathbf{x}^0)(x_i - x_i^0)\right) + f(\mathbf{x}^0) = z$$

As an example, consider the paraboloid, $f(x_1, x_2) = 9 - x_1^2 - x_2^2$, the corresponding $F(x_1, x_2, z) = 9 - x_1^2 - x_2^2 - z$ and the point $x^0 = (\mathbf{x}^0, z) = (1, 1, 7)$ which lies on the 0-level surface of F. The gradient $\nabla F(x_1, x_2, z)$ is $[-2x_1, \ -2x_2, \ -1]$, which when evaluated at $x^0 = (1, 1, 7)$ is $[-2, \ -2, \ -1]$. The equation of the tangent plane to f at x^0 is therefore given by $-2(x_1 - 1) - 2(x_2 - 1) + 7 = z$.

Definition 25 [Local maximum]: *A function f of n variables has a local maximum at $\mathbf{x}^0$ if $\exists \epsilon > 0$ such that $\forall \ ||\mathbf{x} - \mathbf{x}^0|| < \epsilon$. $f(\mathbf{x}) \leq f(\mathbf{x}^0)$. In other words, $f(\mathbf{x}) \leq f(\mathbf{x}^0)$ whenever $\mathbf{x}$ lies in some circular disk around $\mathbf{x}^0$.*

Definition 26 [Local minimum]: *A function f of n variables has a local minimum at $\mathbf{x}^0$ if $\exists \epsilon > 0$ such that $\forall \ ||\mathbf{x} - \mathbf{x}^0|| < \epsilon$. $f(\mathbf{x}) \geq f(\mathbf{x}^0)$. In other words, $f(\mathbf{x}) \geq f(\mathbf{x}^0)$ whenever $\mathbf{x}$ lies in some circular disk around $\mathbf{x}^0$.*

Definition 29 [Global maximum]: *A function f of n variables, with domain $\mathcal{D} \subseteq \Re^n$ has an absolute or global maximum at $\mathbf{x}^0$ if $\forall\ \mathbf{x} \in \mathcal{D}$, $f(\mathbf{x}) \leq f(\mathbf{x}^0)$.*

Definition 30 [Global minimum]: *A function f of n variables, with domain $\mathcal{D} \subseteq \Re^n$ has an absolute or global minimum at $\mathbf{x}^0$ if $\forall\ \mathbf{x} \in \mathcal{D}$, $f(\mathbf{x}) \geq f(\mathbf{x}^0)$.*

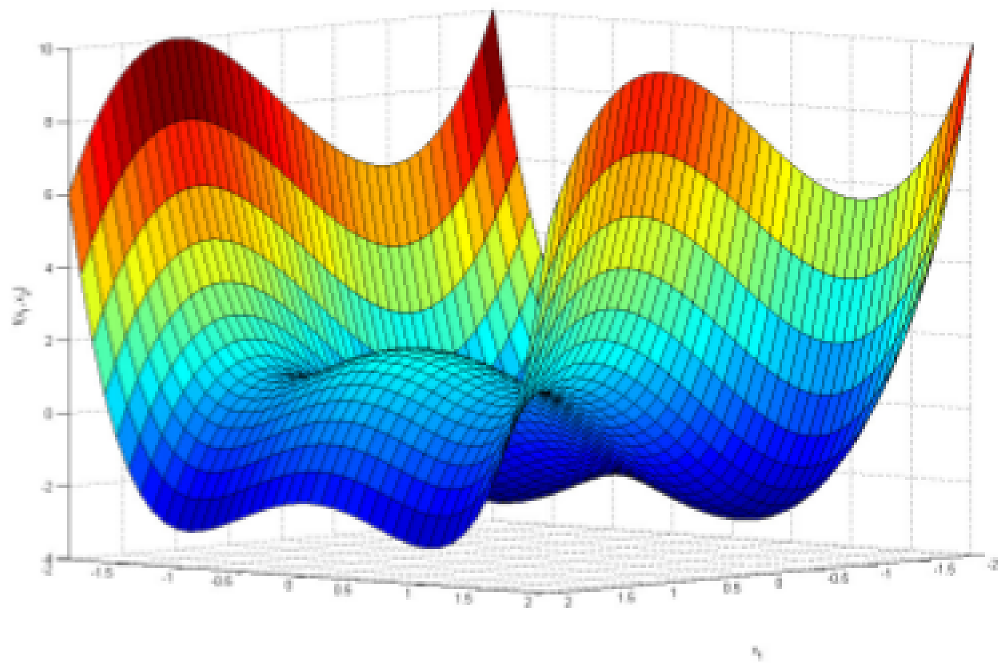

Figure 4.16: Plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$, showing the various local maxima and minima of the function.