Chapter 1

Sets, Relations and Logic

> 'Crime is common. Logic is rare. Therefore it is upon the logic rather than the crime that you should dwell.' Sherlock Holmes in Conan Doyle's *The Copper Breeches*.

1.1 Sets and Relations

1.1.1 Sets

A *set* is a fundamental concept in mathematics. Simply speaking, it consists of some objects, usually called its *elements*. Here are some basic notions about sets that you must already know about:

- A set S with elements a, b and c is usually written as $S = \{a, b, c\}$. The fact that a is an element of S is usually denoted by $a \in S$.
- A set with no elements is called the "empty set" and is denoted by $\emptyset$.
- Two sets S and T are equal ($S = T$) if and only if they contain precisely the same elements. Otherwise $S \neq T$.
- A set T is a subset of a set S ($T \subseteq S$) if and only if every element of T is also an element of S. If $T \subseteq S$ and $S \subseteq T$ then $S = T$. Sometimes $T \subseteq S$ may sometimes also be written as $S \supseteq T$. If $T \subseteq S$ and S has at least one element not in T, then $T \subset S$ (T is said to be "proper subset" of S). Again, $T \subset S$ may sometimes be written as $S \supset T$.

We now look at the meanings of the *union*, *intersection*, and *equivalence* of sets. The intersection, or product, of sets S and T, denoted by $S \cap T$ or ST or $S \cdot T$ consists of all elements common to both S and T. $ST \subset S$ and $ST \subseteq T$ for all sets S and T. Now, if S and T have no elements in common, then they are said to be *disjoint* and $ST = \emptyset$. It should be easy for you to see that $\emptyset \subseteq S$ for all S and $\emptyset \cdot S = \emptyset$ for all S. The union, or sum, of sets S and T, denoted by $S \cup T$ or $S + T$, is the set consisting of elements that belong at least to S or T. Once again, it should be a straightforward matter to see $S \subseteq S + T$ and $T \subseteq S + T$ for all S and T. Also, $S + \emptyset = S$ for all S. Finally, if there is a one-to-one correspondence between the elements of set S and set T, then S and T are said to be equivalent ($S \sim T$). Equivalence and subsets form the basis of the definition of an infinte set: if $T \subset S$ and $S \sim T$ then S is said to be an infinite set. The set of natural numbers $\mathcal{N}$ is an example of an infinite set (any set $S \sim \mathcal{N}$ is said to be *countable* set).

1.1.2 Relations

A finite sequence is simply a set of n elements with a $1-1$ correspondence with the set $\{1, \ldots, n\}$ arranged in order of succession (an *ordered pair*, for example, is just a finite sequence with 2 elements). Finite sequences allow us to formalise the concept of a relation. If A and B are sets, then the set $A \times B$ is called the *cartesian product* of A and B and is denoted by all ordered pairs (a, b) such that $a \in A$ and $b \in B$. Any subset of $A \times B$ is a binary relation, and is called a relation from A to B. If $(a, b) \in R$, then aRb means "a is in relation R to b" or, "relation R holds for the ordered pair (a, b)" or "relation R holds between a and b." A special case arises from binary relations within elements of a single set (that is, subsets of $A \times A$). Such a relation is called a "relation in A" or a "relation over A". There are some important kinds of properties that may hold for a relation R in a set A:

Reflexive. The relation is said to be reflexive if the ordered pair $(a, a) \in R$ for every $a \in A$.

Symmetric. The relation is said to be symmetric if $(a, b) \in R$ *iff* $(b, a) \in R$ for $a, b \in A$.

Transitive. The relation is said to be transitive if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Here are some examples:

- The relation $\leq$ on the set of integers is reflexive and transitive, but not symmetric.

- The relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$ on the set $A = \{1, 2, 3, 4\}$ is reflexive, symmetric and transitive.

- The relation $\div$ on the set $\mathcal{N}$ defined as the set $\{(x, y) : \exists z \in \mathcal{N} \text{ s.t. } xz = y\}$ is symmetric and transitive, but not reflexive.

- The relation $\perp$ on the set of lines in a plane is symmetric but neither reflexive nor transitive.

It should be easy to see a relation like R above is just a set of ordered pairs. Functions are just a special kind of binary relation F which is such that if $(a, b) \in F$ and $(a, c) \in F$ then $b = c$. Our familiar notion of a function F from a set A to a set B is one which associates with each $a \in A$ exactly one element $b \in B$ such that $(a, b) \in F$. Now, a function from a set A to itself is usually called a *unary operation* in A. In a similar manner, a *binary operation* in A is a function from $A \times A$ to A (recall $A \times A$ is the Cartesian product of A with itself: it is sometimes written as A^2). For example, if $A = \mathcal{N}$, then addition ($+$) is a binary operation in A. In general, an n-ary operation F in A is a function from A^n to A, and if it is defined for every element of A^n, then A is said to be *closed* with respect to the operation F. A set which closed for one or more n-ary operations is called an *algebra*, and a sub-algebra is a subset of such a set that remains closed with respect to those operations. For example:

- $\mathcal{N}$ is closed wrt the binary operations of $+$ and $\times$, and $\mathcal{N}$ along with $+, \times$ form an algebra.
- The set $\mathcal{E}$ of even numbers is a subalgebra of algebra of $\mathcal{N}$ with $+, \times$. The set $\mathcal{O}$ of odd numbers is not a subalgebra.
- Let $S \subseteq U$ and $S' \subseteq U$ be the set with elements of U not in S (the unary operation of complementation). Let $U = \{a, b, c, d\}$. The subsets of U with the operations of complementation, intersection and union form an algebra. (How many subalgebras are there of this algebra?)

Equivalence Relations

Any relation R in a set A for which all three properties hold (that is, R is reflexive, symmetric, and transitive) is said to be an "equivalence relation". Suppose, for example, we are looking at the relation R over the set of natural numbers $\mathcal{N}$, which consists of ordered pairs (a, b) such that $a + b$ is even[1] You should be able to verify that R is an equivalence relation over $\mathcal{N}$. In fact, R allows us to split $\mathcal{N}$ into two disjoint subsets: the set of odd numbers $\mathcal{O}$ and the set of even numbers $\mathcal{E}$ such that $\mathcal{N} = \mathcal{O} \cup \mathcal{E}$ and R is an equivalence relation over each of $\mathcal{O}$ and $\mathcal{E}$. This brings us to an important property of equivalence relations:

Theorem 1 *Any equivalence relation E over a set S partitions S into disjoint non-empty subsets $S_1, \ldots, S_k$ such that $S = S_1 \cup \cdots \cup S_k$.*

Let us see how E can be used to partition S by constructing subsets of S in the following way. For every $a \in S$, if $(a, b) \in E$ then a and b are put in the same subset. Let there be k such subsets. Now, since $(a, a) \in E$ for every $a \in S$, every element of S is in some subset. So, $S = S_1 \cup \cdots \cup S_k$. It also follows that the subsets are disjoint. Otherwise there must be some $c \in S_i, S_j$. Clearly, S_i

[1] Equivalence is often denoted by $\approx$. Thus, for an equivalence relation E, if $(a, b) \in E$, then $a \approx b$.

and S_j are not singleton sets. Suppose S_i contains at least a and c. Further let there be a $b \notin S_i$ but $b \in S_j$. Since $a, c \in S_i$, $(a, c) \in E$ and since $c, b \in S_j$, $(c, b) \in E$. Thus, we have $(a, c) \in E$ and $(c, b) \in E$, which must mean that $(a, b) \in E$ (E is transitive). But in this case b must be in the same subset as a by construction of the subsets, which contradicts our assumption that $b \notin S_i$. The converse of this is also true:

Theorem 2 *Any partition of a set S partitions into disjoint non-empty subsets $S_1, \ldots, S_k$ such that $S = S_1 \cup \cdots \cup S_k$ results in an equivalence relation over S.*

(Can you prove that this is the case? Start by constructing a relation E, with $(a, b) \in E$ if and only if a and b are in the same block, and prove that E is an equivalence relation.)

Each of the disjoint subsets $S_1, S_2, \ldots$ are called ”equivalence classes”, and we will denote the equivalence class of an element a in a set S by $[a]$. That is, for an equivalence relation E over a set S:

$$[a] = \{x : x \in S, (a, x) \in E\}$$

What we are saying above is that the collection of all equivalence classes of elements of S forms a partition of S; and conversely, given a partition of the set S, there is an equivalence relation E on S such that the sets in the partition (sometimes also called its ”blocks”) are the equivalence classes of S.

Partial Orders

Given an equality relation $=$ over elements of a set S, a partial order $\preceq$ over S is a relation over S that satisfies the following properties:

Reflexive. For every $a \in S$, $a \preceq a$

Anti-Symmetric. If $a \preceq b$ and $b \preceq a$ then $a = b$

Transitive. If $a \preceq b$ and $b \preceq c$ then $a \preceq c$

Here are some properties about partial orders that you should know (you will be able to understand them immediately if you take, as a special case, $\preceq$ as meaning $\leq$ and $\prec$ as meaning $<$):

- If $a \preceq b$ and $a \neq b$ then $a \prec b$
- $b \succeq a$ means $a \preceq b$, $b \succ a$ means $a \prec b$
- If $a \preceq b$ or $b \preceq a$ then a, b are comparable, otherwise they are not comparable.

A set S over which a relation of partial order is defined is called a *partially ordered set*. It is sometimes convenient to refer to a set S and a relation R defined over S together by the pair $< S, R >$. So, here are some examples of partially ordered sets $< S, \preceq >$: