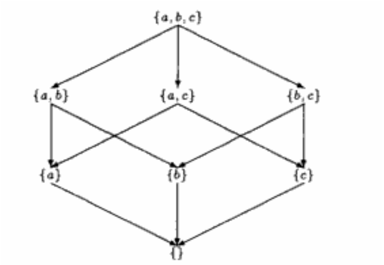

Figure 1.1: The lattice structure of $\langle S, \preceq \rangle$, where S is the power set of $\{a, b, c\}$.

- S is a set of sets, $S_1 \preceq S_2$ means $S_1 \subseteq S_2$
- $S = \mathcal{N}$, $n_1 \preceq n_2$ means $n_1 = n_2$ or there is a $n_3 \in \mathcal{N}$ such that $n_1 + n_3 = n_2$
- S is the set of equivalence relations $E_1, \ldots$ over some set T, $E_L \preceq E_M$ means for $u, v \in T$, uE_Lv means uE_Mv (that is, $(u, v) \in E_L$ means $(u, v) \in E_M$).

Given a set $S = \{a, b, \ldots\}$ if $a \prec b$ and there is no $x \in S$ such that $a \prec x \prec b$ then we will say b *covers* a or that a is a *downward cover* of b. Now, suppose S_{down} be a set of downward covers of $b \in S$. If for all $x \in S$, $x \prec b$ implies there is an $a \in S_{down}$ s.t. $x \preceq a \prec b$, then S_{down} is said to be a *complete* set of downward covers of b. Partially ordered sets are usually shown as diagrams like in Figure 1.1.
The diagrams, as you can see, are graphs (sometimes called Hasse graphs or Hasse diagrams). In the graph, vertices represent elements of the partially ordered set. A vertex v_2 is at a higher level than vertex v_1 whenever $v_1 \prec v_2$, and there is an edge between the two vertices only if v_2 covers v_1 (that is, v_2 is an immediate predecessor). The graph is therefore really a directed one, in which there is a directed edge from a vertex v_2 to v_1 whenever v_2 covers v_1. Also, since the relation is anti-symmetric, there can be no cycles. So, the graph is a directed acyclic graph, or DAG.

In the diagram in Figure **??** on the left, S is the set of non-empty subsets of $\{a, b, c\}$ and $\preceq$ denotes the subset relationship (that is, $S_1 \preceq S_2$ if and only if $S_1 \subset S_2$). The diagram on the right is an example of a *chain*, or a *totally ordered* set.

You should be able to see that a finite chain of length n can be put in a one-to-one correspondence to a finite sequence of natural numbers $(1, \ldots, n)$ (the correct way to say this is that a finite chain is isomorphic with a finite sequence of natural numbers). In general, a partially ordered set S is a chain if for every pair $a, b \in S$, $a \prec b$ or $b \prec a$. There is a close relationship between a partially ordered set and a chain. Suppose S is a partially ordered set. We can always associate a function f from the elements of S to $\mathcal{N}$ (the set of natural numbers), so that if $a \prec b$ for $a, b \in S$, then $f(a) < f(b)$. f is called a *consistent enumeration* of S, and is not unique and we can use it to define a chain consistent with S. (We will leave the proof of the existence a consistent enumeration for you. One way would be to use the method of induction on the number of elements in S: clearly there is such an enumeration for $|S| = 1$. Assume that an enumeration exists for $|S| = n - 1$ and prove it for $|S| = n$.)

Some elements of a partially ordered set have special properties. Let $< S, \preceq >$ be a p.o. set and $T \subseteq S$. Then (in the following, you should read the symbol $\exists$ as being shorthand for "there exists", and $\forall$ as "for all"):

– Least element of T $a \in T \ s.t. \ \forall t \in T \ a \preceq t$	– Greatest element of T $a \in T s.t. \ \forall t \in T \ a \succeq t$
– Least element, if it exists, is unique. If $T = S$ this is the "zero" element	– Greatest element, if it exists is unique. If $T = S$ then this is the "unity" element
– Minimal element of T $a \in T \ \ \nexists t \in T \ s.t. \ t \prec a$	– Maximal element of T $a \in T \ \ \nexists t \in T \ s.t. \ t \succ a$
– Minimal element need not be unique	– Maximal element need not be unique
– Lower bound of T $b \in S \ s.t. \ b \preceq t \ \forall t \in T$	– Upper bound of T $b \in S \ s.t. \ b \succeq t \ \forall t \in T$
– Glb g of T $b \preceq g \ \forall b, g: \ lbs \ of \ T$	– Lub g of T $b \succeq g \ \forall b, g: \ ubs \ of \ T$
– If it exists, the glb is unique	– If it exists the lub is unique

As you would have observed, there is a difference between a least element and a minimal element (and correspondingly, between greatest and maximal elements). The requirement of a minimal (maximal) upper bound is, in some sense, a weakening of the requirement of a least (greatest) upper bound. If x and y are both lub's of some set $T \subseteq S$, then $y \preceq x$ and $z \preceq y$, so then $x \approx y$. This means that all lub's of T are equivalent. Dually, if x and y are glb's of some T, then also $x \approx y$. Thus, if a least element exists, then it is unique: this is not necessarily the case with a minimal element. Also, least and greatest elements must belong to the set T, but lower and upper bounds need not.

For this example, S has: (1) one upper bound b; (2) no lower bound; (3) a greatest element b; (4) no least element; (5) no greatest lower bound; (6) two minimal elements a and e; and (7) one maximal element b. Can you identify what the corresponding statements are for T?

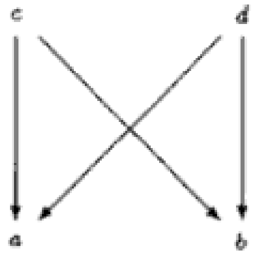

Figure 1.2: $\{a, b\}$ has no lub here.

The glb and lub are sometimes also taken to be binary operations on a partially ordered set S, that assigns to an ordered pair in S^2 the corresponding glb or lub. The first operation is called the *product* or *meet* and is denoted by $\cdot$ or $\sqcap$. The second operation is sometimes called the *sum* or *join* and is denoted by $+$ or $\sqcup$.

In a quasi-ordered set, a subset need not have a lub or glb. We will take an example to illustrate this. Let $S = \{a, b, c, d\}$, and let $\preceq$ be defined as $a \preceq c$, $b \preceq c$, $a \preceq d$ and $d \preceq b$. Then since c and d are incomparable, the set a, b has no lub in this quasi-order. See Figure 1.2.

Similarly, a set need not have a maximal or a minimal, nor upward or downward covers. For instance, let S be the infinite set $\{y, x_l, x_2, x_3, \ldots\}$, and let $\preceq$ be a quasi-order on S, defined as $y \prec \ldots x_{n+1} \prec x_n \prec \ldots \prec x_2 \prec x_1$. Then there is no upward cover of y: for every x_n, there always is an x_{n+l} such that $y \prec x_{n+1} \prec x_n$. In this case, y has no complete set of upward covers.

Note that a complete set of upward covers for y need not contain all upward covers of y. However, in order to be complete, it should contain at least one element from each equivalence class of upward covers. On the other hand, even the set of all upward covers of y need not be complete for y. For the example given above, the set of all upward covers of y is empty, but obviously not complete.

A notion of some relevance later is that of a function f defined on a partially ordered set $< S, \preceq >$. Specifically, we would like to know if the function is: (a) monotonic; and (b) continuous. Monotonicity first:

> A function f on $< S, \preceq >$ is monotonic if and only if for all $u, v \in S$, $u \preceq v$ means $f(u) \preceq f(v)$

Now, suppose a subset S_1 of S have a least upper bound $lub(S_1)$ (with some abuse of notation: here $lub(X)$ is taken to be the lub of the elements in set X). Such subsets are called "directed" subsets of S. Then:

> A function f on $< S, \preceq >$ is continuous if and only if for all directed subsets S_i of S, $f(lub(S_i)) = lub(\{f(x) : x \in S_i\})$.

That is, if a directed set S_i has a least upper bound $lub(S_i)$, then the set obtained by applying a continuous function f to the elements of S_i has least upper bound $f(lub(S_i))$. Functions that are both monotonic and continuous on some partially ordered set $< S, \preceq >$ are of interest to us because they can be used, for some kinds of orderings, to guarantee that for some $s \in S$, $f(s) = s$. That is, f is said to have a "fixpoint".

Lattices

A lattice is just a partially ordered set $< S, \preceq >$ in which every pair of elements $a, b \in S$ has a glb (represented by $\sqcap$) and a lub (represented by $\sqcup$). From the definitions of lower and upper bounds, we are able to show that in any such partially ordered set, the operations will have the following properties:

- $a \sqcap b = b \sqcap a$, and $a \sqcup b = b \sqcup a$ (that is, they are are commutative).
- $a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$, and $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$ (that is, they are associative).
- $a \sqcap (a \sqcup b) = a$, and $a \sqcup (a \sqcap b) = a$ (that is, they are "absorptive").
- $a \sqcap b = a$ and $a \sqcup b = b$.

We will not go into all the proofs here, but show one for illustration. Since $a \sqcap b$ is the glb of a and b, $a \sqcap b \preceq a$. Clearly then $a \sqcup (a \sqcap b)$, which is the lub of a and $a \sqcap b$, is a. This is one of the absorptive properties above. You should also be able to see, from these properties, that a lattice can also be seen simply as an algebra with two binary operations $\sqcap$ and $\sqcup$ that are commutative, associative and absorptive.

Theorem 3 *A lattice is an algebra with the binary operations of* $\sqcup$ *and* $\sqcap$.

Here is an example of a lattice: let S be all the subsets of $\{a, b, c\}$, and for $X, Y \in S$, $X \preceq Y$ means $X \subseteq Y$, $X \sqcap Y = X \cap Y$ and $X \sqcup Y = X \cup Y$. Then $< S, \subseteq >$ is a lattice. The empty set $\emptyset$ is the zero element, and S is the unity element of the lattice. More generally, a lattice that has a zero or least element (which we will denote $\perp$), and a unity or greatest element (which we will denote $\top$) is called a *bounded* lattice. In such lattices, the following necessarily hold: $a \sqcup \top = \top$; $a \sqcap \top = a$; $a \sqcup \perp = a$; and $a \sqcap \perp = \perp$. A little thought should convince you that a finite lattice will always be bounded: if the lattice is the set $S = \{a_1, \ldots, a_n\}$ then $\top = a_1 \sqcup \cdots \sqcup a_n$ and $\perp = a_1 \sqcap \cdots \sqcap a_n$. (But, does the reverse hold: will a bounded lattice always be finite?)

Two properties of subsets of lattices are of interest to us. First, a subset M of a lattice L is called a *sublattice* of L if M is also closed under the same binary operations of $\sqcup$ and $\sqcap$ defined for L (that is, M is a lattice with the same operations as those of L). Second, if a lattice L has the property that every subset of L has a lub and a glb, then the L is said to be a *complete* lattice. Clearly, every finite lattice is complete. Further, since every subset of