$\mathbf{w} \succeq \mathbf{l}_2$. Thus there are substitutions γ, μ, ν, such that $\mathbf{u}\gamma = \mathbf{w}$, $\mathbf{l}_1 = \mathbf{w}\mu = \mathbf{u}\gamma\mu$ and $\mathbf{l}_2 = \mathbf{w}\nu = \mathbf{u}\gamma\nu$. Then $\mathbf{u}\theta = \mathbf{l}_1 = \mathbf{u}\gamma\mu$ and $\mathbf{u}\sigma = \mathbf{l}_2 = \mathbf{u}\gamma\nu$ (see Figure 1.13 for illustration). Hence, if x is a variable occurring in $\mathbf{u}$, then $x\theta = x\gamma\mu$ and $x\sigma = x\gamma\nu$.

We will now show that $\mathbf{u}$ and $\mathbf{w} = \mathbf{u}\gamma$ are variants, by showing that γ is a renaming substitution for $\mathbf{u}$. Suppose it is not. Then γ maps some variable x in $\mathbf{u}$ to a term that is not a variable, or γ unifies two distinct variables x, y in $\mathbf{u}$.

Suppose x is a variable in $\mathbf{u}$, such that $x\gamma = t$, where t is a term that is not a variable. If x is not one of the z_j's, then $x\gamma\mu = x\theta = x$, contradicting the assumption that $x\gamma = t$ is not a variable. But on the other hand, if x equals some z_j, then $t\mu = x\gamma\mu = x\theta = s_j$ and $t\nu = x\gamma\nu = x\sigma = t_j$. Then s_j and t_j would both start with the first symbol of t, contradicting theorem 21, part 3. So this case leads to a contradiction.

Suppose x and y are distinct variables in $\mathbf{u}$ such that γ unifies x and y. Then,

1. If neither x nor y is one of the z_j's, then $x\gamma\mu = x\theta = x \neq y = y\theta = y\gamma\mu$, contradicting $x\gamma = y\gamma$
2. If x equals some z_j and y does not, then $x\gamma\mu = x\theta = s_j$ and $x\gamma\nu = x\sigma = t_j$, so $x\gamma\mu \neq x\gamma\nu$ by theorem 21, part 3. But $y\gamma\mu = y\theta = y = y\sigma = y\gamma\nu$, contradicting $x\gamma = y\gamma$.
3. Similarly for the case where y equals some z_j and x does not.
4. If $x = z_j$ and $y = z_k$, then $j \neq k$, since $x \neq y$. Furthermore, $s_j = x\theta = x\gamma\mu = y\gamma\mu = y\theta = s_k$ and $t_j = x\sigma = x\gamma\nu = y\gamma\nu = y\sigma = t_k$. But this contradicts theorem 21, part 4.

Thus, the assumption that γ unifies two variables in $\mathbf{u}$ also leads to a contradiction. Thus γ is a renaming substitution for $\mathbf{u}$, and hence $\mathbf{u}$ and $\mathbf{w}$ are variants. Finally, since $\mathbf{v} \succeq \mathbf{w}$, we have $\mathbf{v} \succeq \mathbf{u}$. $\square$

It is however a more straightforward matter to see the following:

Theorem 23 *If $\mathcal{A}_E^+$ is a set of equivalance classes of atoms $\mathcal{A}$ augmented by the two elements $\top$ and $\bot$, then for any pair of elements $\mathbf{l}_1, \mathbf{l}_2 \in \mathcal{A}_E^+$, $\mathbf{l}_1 \sqcup \mathbf{l}_2$ always exists.*

Proof: The possibilities for each of $\mathbf{l}_1$ and $\mathbf{l}_2$ are that they are either: (1) some variant of $\top$; (2) some variant of $\bot$; or (3) an atom from S. It can be verified that $\mathbf{l}_1 \sqcap \mathbf{l}_2$ is defined in all 9 cases.

1. If $\mathbf{l}_1 = \top$ or $\mathbf{l}_2 = \top$, then $\mathbf{l}_1 \sqcup \mathbf{l}_2 = \top$. If $\mathbf{l}_1 = \bot$, then $\mathbf{l}_1 \sqcup \mathbf{l}_2 = \mathbf{l}_2$. If $\mathbf{l}_2 = \bot$, then $\mathbf{l}_1 \sqcup \mathbf{l}_2 = \mathbf{l}_1$.
2. If $\mathbf{l}_1$ and $\mathbf{l}_2$ are conventional atoms with the same predicate symbol, $\mathbf{l}_1 \sqcup \mathbf{l}_2$ is given by the Anti-Unification Algorithm.

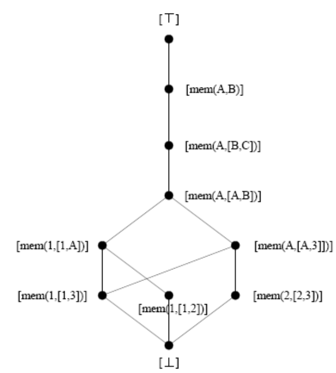

Figure 1.14: An example subsumption lattice over atoms.

3. If $\mathbf{l}_1$ and $\mathbf{l}_2$ are conventional atoms with different predicate symbols, then $\mathbf{l}_1 \sqcup \mathbf{l}_2 = \top$.

□

Now that we have established the existence of an lub and glb of any $\mathbf{l}_1, \mathbf{l}_2 \in \mathcal{A}_E^+$, we have shown that the set of atoms ordered by subsumption, is a lattice.

Theorem 24 *Let $\mathcal{A}$ be the set of atoms. Then $< \mathcal{A}_E^+, \succeq >$ is a lattice.*

Figure 1.14 shows an example subsumption lattice over atoms in $S^+ = \{ \top, \perp, mem(1,[1,3]), mem(1,[1,2]), mem(2,[2,3]), mem(1,[1,A]), mem(A,[A,B]), mem(A,[A,3]), mem(A,[B,C]), mem(A,[B|C])\ mem(A,B)\ \}$ Note that

- $l = mem(A,[A,B]) \succeq mem(1,[1,2]) = m$ since with $\theta = \{A/1, B/2\}$, $l\theta = m$
- $mem(A1,[A1,B1]), mem(A2,[A2,B2]) \ldots$ are all members of the same equivalence class

Recall that, for atoms $l, m \in S$, subsumption is equivalent to implication. That is, if $l \models m$ then $l \succeq m$. Least-general-generalisation of atoms will be enountered in Lab Nos. 5, 6.

What about subsumption over clauses with more than just one literal? Is this still a quasi-order, with a lattice structure over equivalence classes of clauses? The short answer is "yes", but more on this in Chapter 2.

1.4.8 Covers of Atoms

What about the covers relation in the subsumption lattice of atoms? Recall that covers are the smallest non-trivial steps between individual atoms that we can take in the lattice. Since $\mathbf{l}_2$ is a downward cover of $\mathbf{l}_1$ iff $\mathbf{l}_1$ is an upward cover of $\mathbf{l}_2$, we will first restrict attention to downward covers.

Downward Covers

Theorem 25 *Let* $\mathbf{l}_1$ *be a conventional atom,* f *an* n*-ary function symbol (recall that* f *can be of zero arity and therefore a constant),* z *a variable in* $\mathbf{l}_1$*, and* $x_1, \ldots, x_n$*, distinct variables not appearing in* $\mathbf{l}_1$*. Let*

1. $\theta = \{z/f(x_1, \ldots, x_n)\}$ *and*
2. $\sigma = \{x_i/x_j\}$, $i \neq j$

Then $\mathbf{l}_2 = \mathbf{l}_1\theta$ *and* $\mathbf{l}_3 = \mathbf{l}_1\sigma$ *are both downward covers of* $\mathbf{l}_1$*. In fact, every downward cover of* $\mathbf{l}_1$ *must of one of these two forms (note that a special instance of the first case is when the function* f *has arity* 0 *and is therefore a constant). The substitutions* θ *and* σ *are termed elementary substitutions. In ILP, these 3 operations define a "downward refinement operator"*

Proof:

Proof for (1): It is clear that $\mathbf{l}_1$ and $\mathbf{l}_2$ are not variants, so $\mathbf{l}_1 \succ \mathbf{l}_2$. Suppose there is a $\mathbf{m}$ such that $\mathbf{l}_1 \succ \mathbf{m}\mathbf{l}_2$. Then there are γ, μ, such that $\mathbf{l}_1\gamma = \mathbf{m}$ and $\mathbf{m}\mu = \mathbf{l}_2$, hence $\mathbf{l}_1\gamma\mu = \mathbf{l}_2 = \mathbf{l}_1\theta$. Here γ only acts on variables in $\mathbf{l}_1$, and μ only acts on variables in $\mathbf{m}$.

Let (x, p) be a term occurrence in $\mathbf{l}_1$, where x is a variable. Suppose $x \neq z$, then $x\theta = x$, so (x, p) must also be a term occurrence in $\mathbf{l}_2$. Hence $x\gamma$ must be a variable, for otherwise $(x\gamma\mu, p)$ in $\mathbf{l}_2$ would contain a constant or a function. Thus γ must map all variables other than z to variables. Furthermore, γ cannot unify two distinct variables in $\mathbf{l}_1$, for then $\mathbf{l}_2$ would also have to unify these two variables, which is not the case.

If $z\gamma$ is also a variable, then γ would map all variables to variables, and since γ cannot unify distinct variables, it would map all distinct variables in $\mathbf{l}_1$ to distinct variables. But then γ would be a renaming substitution for $\mathbf{l}_1$, contradicting $\mathbf{l}_1 \succ \mathbf{m}$. Hence γ must map z to some term containing a function symbol.

Now the only way we can have $\mathbf{l}_1\gamma\mu = \mathbf{l}_2$, is if $z\gamma = f(y_1, \ldots, y_n)$ for distinct y_i not appearing in $\mathbf{l}_1$, and no variable in $\mathbf{l}_1$ is mapped to some y_i by γ. But then $\mathbf{l}_1\gamma$ and $\mathbf{l}_2$ would be variants, contradicting $\mathbf{l}_1\gamma = \mathbf{m} \succ \mathbf{l}_2$. Therefore such a $\mathbf{m}$ does not exist, and $\mathbf{l}_2$ is a downward cover of $\mathbf{l}_1$.

Proof for (2): It is clear that $\mathbf{l}_1 \succ \mathbf{l}_3$. Suppose there is a $\mathbf{m}$ such that $\mathbf{l}_1 \succ \mathbf{m} \succ \mathbf{l}_3$. Then there are γ, μ, such that $\mathbf{l}_1\gamma = \mathbf{m}$ and $\mathbf{m}\mu = \mathbf{l}_3$, hence $\mathbf{l}_1\gamma\mu = \mathbf{l}_3 = \mathbf{l}_1\sigma$. Here γ only acts on variables in $\mathbf{l}_1$, and μ only on variables in $\mathbf{m}$. Note that γ and μ can only map variables to variables, since otherwise $\mathbf{l}_1\gamma\mu = \mathbf{l}_3$ would contain more occurrences of functions or constants than $\mathbf{l}_1$, contradicting $\mathbf{l}_1\sigma = \mathbf{l}_3$, since σ does not add any occurrences of function symbols to $\mathbf{l}_1$.

If γ does not unify any variables in $\mathbf{l}_1$, then $\mathbf{l}_1$ and $\mathbf{m}$ would be variants, contradicting $\mathbf{l}_1 \succ \mathbf{m}$. If γ unifies any other variables than z and x, then we could not have $\mathbf{l}_1\gamma\mu = \mathbf{l}_3$. Hence γ must unify z and x, and cannot unify any other variables. But then $\mathbf{l}_1\gamma$ and $\mathbf{l}_3$ would be variants, contradicting $\mathbf{l}_1\gamma = \mathbf{m} \succ \mathbf{l}_3$. Therefore such a $\mathbf{m}$ does not exist, and $\mathbf{l}_3$ is a downward cover of $\mathbf{l}_1$ $\Box$

Application of the elementary substitutions on $\top$ results in its downward covers called *most general atoms*, that consist of all n-ary predicate symbols, each with n-distinct variables as arguments. Dually, $\mathbf{l}_2$ is an upward cover of $\mathbf{l}_1$ iff $\mathbf{l}_1$ is a downward cover of $\mathbf{l}_2$. Thus the upward covers of some conventional atom $\mathbf{l}_1$ are also of two types, which can be constructed by inverting the two elementary substitutions discussed in theorem 25. Trivially, every ground atom[29] is an upward cover of $\perp$.

Further, it can be proved that given two atoms $\mathbf{l}_1$ and $\mathbf{l}_2$ such that $\mathbf{l}_1 \succ \mathbf{l}_2$ ($\mathbf{l}_2 \succ \mathbf{l}_1$), there is a finite sequence of downward (upward) covers from $\mathbf{l}_1$ ($\mathbf{l}_2$) to a variant of $\mathbf{l}_2$ ($\mathbf{l}_1$). This means that if we want to get from $\mathbf{l}_1$ to (a variant of) $\mathbf{l}_2$, we only need to consider downward (upward) covers of $\mathbf{l}_1$, downward (upward) covers of downward (upward) covers of $\mathbf{l}_1$, *etc.* In fact, there is a *finite downward cover chain algorithm* for this purpose, which is outlined in Figure 1.15.

INPUT: Conventional atoms $\mathbf{l}, \mathbf{m}$, such that $\mathbf{l} \succ \mathbf{m}$.
OUTPUT: A finite chain $\mathbf{l} = \mathbf{l}_0 \succ \mathbf{l}_1 \succ \ldots \succ \mathbf{l}_{n-1} \succ \mathbf{l}_n = \mathbf{m}$, where each $\mathbf{l}_{i+1}$ is a downward cover of $\mathbf{l}_i$.
Set $\mathbf{l}_0 = \mathbf{l}$ and $i = 0$, let θ_0 be such that $\mathbf{l}\theta_0 = \mathbf{m}$; **(1)**
if No term in θ_i contains a function or a constant; **(2) then**
  Goto 3.
else if $x/f(t_1, \ldots, t_n)$ is a binding in θ_i ($n \geq 0$) **then**
  Choose new distinct variables $z_l, \ldots, z_n$;
  Set $\mathbf{l}_{i+1} = \mathbf{l}_i\{x/f(z_1, \ldots, z_n)\}$;
  Set $\theta_{i+1} = (\theta_i \setminus \{x/f(t_1, \ldots, t_n)\}) \cup \{z_1/t_1, \ldots, z_n/t_n\}$;
  Set i to $i + 1$ and goto 2;
end if
if There are distinct variables x, y in $\mathbf{l}_i$, such that $x\theta_i = y\theta_i$ **(3) then**
  Set $\mathbf{l}_{i+1} = \mathbf{l}_i\{x/y\}$;
  Set $\theta_{i+1} = \theta_i \setminus \{x/x\theta_i\}$;
  Set i to $i + 1$ and goto 3;
else if Such x, y do not exist **then**
  Set $n = i$ and stop;
end if

Figure 1.15: Finite Downward Cover Chain Algorithm.

The subsumption ordering on atoms can be summarized through the following example:

- $l = mem(A, [A, B]) \succeq mem(1, [1, 2]) = m$ since with $\theta = \{A/1, B/2\}$, $l\theta = m$
- $mem(A1, [A1, B1]), mem(A2, [A2, B2]) \ldots$ are all members of the same equivalence class

[29] Number of ground atoms can be infinite if the language consists of a function symbol of arity ≥ 1.

Upward Covers

To construct a finite chain of upward covers to $\mathbf{l}_1$, starting from $\mathbf{l}_2$, where $\mathbf{l}_1 \succ \mathbf{l}_2$, the algorithm 1.15 needs to be reversed. While algorithm 1.15 first instantiates variables to functions and constants, and then unifies some variables, the reverse algorithm "undoes" unifications and instantiations using *inverse substitution* (which is, strictly speaking, not a function).

One asymmetry of the downward and upward cases concerns the upward covers of $\perp$. In case of a language without constants but with at least one function symbol of arity ≥ 1, the bottom element $\perp$ has no upward covers at all, let alone a finite complete set of upward covers. In case of a language with at least one constant and at least one function symbol of arity ≥ 1, there are an infinite number of conventional ground atoms, each of which is an upward cover of $\perp$. Together these ground atoms comprise a complete set of upward covers of $\perp$, but again $\perp$ has no finite complete set of upward covers in this case. However, each conventional atom does have a finite complete set of upward covers. The top element $\top$ does not have any upward covers at all, but it has the empty set as a finite complete set of upward covers, since no element lies "above" $\top$.

1.4.9 The Subsumption Theorem Again

The Subsumption Theorem holds for first-order logic, just as it did for propositional logic:

> If Σ is a set of first-order clauses and D is a first-order clause. Then $\Sigma \models D$ if and only if D is a tautology or there is a clause C such that there is a derivation of C from Σ using resolution ($\Sigma \vdash_R C$) and C subsumes D.

By "derivation of a clause C" here, we mean the same as in propositional logic (page 30), that is, there is a sequence of clauses $R_1, \ldots, R_k = C$ such that each R_i is either in Σ or is a resolvent of a pair of clauses in $\{R_1, \ldots, R_{i-1}\}$. While extending the proof of theorem 9, the proof of this is a bit involved and we do not present it here: we refer you to [NCdW97] for a complete proof that shows that the result does indeed hold.

An immediate consequence is that the refutation-completeness of resolution follows for first-order logic as well. It is not possible, therefore, to decide, using resolution, whether a set Σ of Horn clauses is, in fact, unsatisfiable (that is, $\Sigma \models \Box$: in fact, we will see later that the roots of this undecidability is more fundamental than just to do with Horn clauses or resolution).[30] All that we are saying with refutation-completeness is that <u>if</u> $\Sigma \models \Box$ then there is a resolution

[30] This gives us another difference between implication and subsumption. Unlike implication, subsumption between a pair of clauses *is* decidable, although not necessarily efficiently in all cases. We can informally show that it is decidable whether a clause C subsumes a clause D. If $C \succeq D$, then there is a substitution θ which maps each $\mathbf{l}_i \in C$ to some $\mathbf{l}_j \in D$. If C contains n literals, and D contains m literals, then there are m^n ways in which the literals in C can be paired up with literals in D. Then we can decide $C \succeq D$ by checking whether for at least one of those m^n ways of pairing the n literals in C to some of the m literals in D, there is a