L has a lub and a glb, this must certainly be true of L itself. So, L has a lub, which must necessarily be the greatest element of L. Similarly, L has a glb, which must necessarily be the least element of L. In fact, the elements of L are ordered in such a way that each element is on some path from $\top$ to $\bot$ in the Hasse diagram. An example of an ordered set that is always a complete lattice is the set of all subsets of a set S, ordered by $\subseteq$, with binary operations $\cap$ and $\cup$ for the glb and lub. This set, the "powerset" of S, is often denoted by 2^S. So, if $S = \{a, b, c\}$, 2^S is the set $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Clearly, every subset of s of 2^S has both a glb and a lub in S.

There are two important results concerning complete lattices and functions defined on them. The Knaster-Tarski Theorem tells us that every monotonic function on a complete lattice $< S, \preceq >$ has a least fixpoint.

Theorem 4 *Let $< S, \preceq >$ be a complete lattice and let $f : S \rightarrow S$ be a monotonic function. Then the set of fixed points of f in L is also a complete lattice $< P, \preceq >$ (which obviously means that f has a greatest as well as a least fixpoint).*

Proof Sketch:[2] Let $D = \{x | x \preceq f(x)\}$. From the very definition of D, it follows that every fixpoint is in D. Consider some $x \in D$. Then because f is monotone we have $f(x) \preceq f(f(x))$. Thus,

$$\forall\, x \in D,\ f(x) \in D \tag{1.1}$$

Let $u = lub(D)$ (which should exist according to our assumption that $< S, \preceq >$ is a complete lattice. Then $x \preceq u$ and $f(x) \preceq f(u)$, so $x \preceq f(x) \preceq f(u)$. Therefore $f(u)$ is an upper bound of D. However, u is the least upper bound, hence $u \preceq f(u)$, which in turn implies that, $u \in D$. From (1.1), it follows that $f(u) \in D$. From $u = lub(D)$, $f(u) \in D$ and $u \preceq f(u)$, it follows that $f(u) = u$. Because every fixpoint is in D we have that u is the greatest fixpoint of f. Similarly, it can be proved that if $E = \{x | f(x) \preceq x\}$, then $v = glb(E)$ is a fixed point and therefore the smallest fixpoint of f. $\Box$

Kleene's First Recursion Theorem tells us how to find the element $s \in S$ that is the least fixpoint, by incrementally constructing lubs starting from applying a continuous function to the least element of the lattice ($\bot$).

Theorem 5 *Let S be a complete partial order and let $f : S \rightarrow S$ be a continuous (and therefore monotone) function. Then the least fixed point of f is the supremum of the ascending Kleene chain of f:*

$$\bot \preceq f(\bot) \preceq f(f(\bot)) \preceq \ldots \preceq f^n(\bot) \preceq \ldots$$

In the special case that $\preceq$ is $\subseteq$, the incremental procedure starts with the empty set $\emptyset$, and progressive lub's are obtained by application of the set-union operation $\cup$. We will not give the proofs of this result here.

[2] Can you complete the proof?

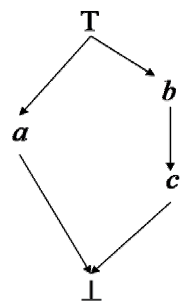

Figure 1.3: Example lattice for illustrating the concept of lattice length.

A final concept we will need is the concept of the length of a lattice. For a pair of elements a, b in a lattice L such that $a \preceq b$, the interval $[a, b]$ is the set $\{x : x \in L, \quad a \preceq x \preceq b\}$. Now, consider a subset of $[a, b]$ that contains both a and b, and is such that any pair of elements in the subset are comparable. Then that subset is a chain from a to b: if the number of elements in the subset is n, then the length of the chain is $n - 1$. Maximal chains from a to b are those of the form $a = x_1 \prec x_2 \prec \cdots \prec x_n = b$ such that each x_i is covered by x_{i+1}. If all maximal chains from a to b are finite, then the longest of these defines the length of the interval $[a, b]$. For a bounded lattice, the length of the interval $[\bot, \top]$ defines the length of the lattice. So, in the lattice in Figure 1.3, there are two maximal chains between $\bot$ and $\top$, of lengths 2 and 3 (what are these?). The length of lattice is thus equal to 3. Now, it should be evident that finite lattices will always have a finite length, but it is possible for lattices to have a finite length, but have infinitely many elements. For example, the lattice $L = \{\bot, \top, x_1, x_2, \ldots\}$ such that $\bot \prec x_i \prec \top$ has a finite length (all maximal chains are of length 2). (Indeed, it is even possible to have an infinite set in which maximal chains are of finite, but increasing in lengths of $1, 2 \ldots$.)

Quasi-Orders

A quasi-order Q in a set S is a binary relation over S that satisfies the following properties:

Reflexive. For every $a \in S$, aQa

Transitive. If aQb and bQc then aQc

You can see that a quasi-order differs from an equivalence relation in that symmetry is not required. Further, it differs from a partial order because no equality is defined, and therefore the property of anti-symmetry property cannot be defined either. There are two important properties of quasi-orders, which we will present here without proof:

- If a quasi-order Q is defined on a set $S = \{a, b, \ldots\}$, and we define a binary relation E as follows: aEb iff aQb and bQa. Then E is an equivalence relation.

- Let E partition S into subsets $X, Y, \ldots$ of equivalent elements. Let $T = \{X, Y, \ldots\}$ and $\preceq$ be a binary relation in T meaning $X \preceq Y$ in T if and only if xQy in S for some $x \in X, y \in Y$. Then T is partially ordered by $\preceq$.

What these two properties say is simply this:

> A quasi-order order Q over a set S results in a partial ordering over a set of equivalence classes of elements in S.

1.2 Logic

Logic, the study of arguments and 'correct reasoning', has been with us for at least the better part of two thousand years. In Greece, we associate its origins with Aristotle (384 B.C.–322 B.C.); in India with Gautama and the Nyaya school of philosophy (3$^{\text{rd}}$ Century B.C.?); and in China with Mo Ti (479 B.C.–381 B.C.) who started the Mohist school. Most of this dealt with the use and manipulation of *syllogisms*. It would only be a small injustice to say that little progress was made until Gottfried Wilhelm von Leibniz (1646–1716). He made a significant advance in the use of logic in mathematics by introducing symbols to represent statements and relations. Leibniz hoped to reduce all errors in human reasoning to minor calculational mistakes. Later, George Boole (1815–1864) established the connection between logic and sets, forming the basis of *Boolean algebra*. This link was developed further by John Venn (1834–1923) and Augustus de Morgan (1806–1872). It was around this time that Charles Dodgson (1832–1898), writing under the pseudonym Lewis Carroll, wrote a number of popular logic textbooks. Fundamental changes in logic were brought about by Friedrich Ludwig Gottlob Frege (1848–1925), who strongly rejected the idea that the laws of logic are synonymous with the laws of thought. For Frege, the former were laws of *truth*, having little to say on the processes by which human beings represent and reason with reality. Frege developed a logical framework that incorporated propositions with relations and the validity of arguments depended on the relations involved. Frege also introduced the device of *quantifiers* and *bound variables*, thus laying the basis for *predicate logic*, which forms the basis of all modern logical systems. All this and more is described by Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947) in their monumental work, *Principia Mathematica*. And then in 1931, Kurt Gödel (1906–1978) showed much to the dismay of mathematicians everywhere that formal systems of arithmetic would remain incomplete.

Rational agents require knowledge of their world in order to make rational decisions. With the help of a declarative (knowledge representation) language, this knowledge (or a portion of it) is represented and stored in a knowledge