Solid Sweeps in CAGD

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joint work with **Bharat Adsul** and **Jinesh Machchhar**

Solid Sweep

Given a solid M and a one parameter family of rigid motions h, compute the volume $\mathcal{V}$ swept by M.

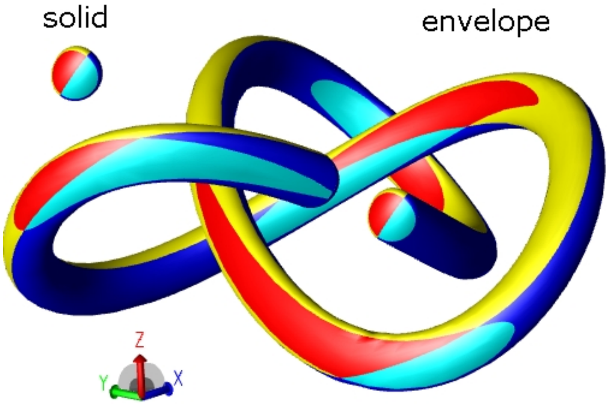


Figure: A solid swept along a trefoil knot.

Applications

- CNC-machining verification
- Collision detection
- Robot path planning
- Machine assembly planning
- Packaging and product handling

Application in product handling: Conveyor screws

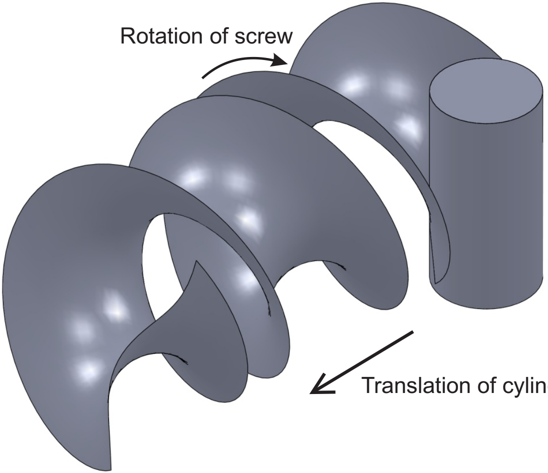


Figure: Conveyor screw.

Application in product handling: Gravity chutes

Figure: Gravity chute. Source: SigmaPackaging.com

Previous work

All previous approaches assume that the input solid is either given implicitly or as a single parametric surface.

- Sweep envelope differential equation (SEDE): Gives an approximation of the envelope.
- Trimming swept volumes: Uses inverse trajectory for trimming over the SEDE framework. Computationally expensive.
- Jacobian rank deficiency condition: Cannot handle free-form surfaces as input.
- Point membership classification: Yields a procedural implicit definition of the envelope. Computationally expensive.

Solid Sweep

Given a solid M and a one parameter family of rigid motions h, **compute** the volume $\mathcal{V}$ swept by M.

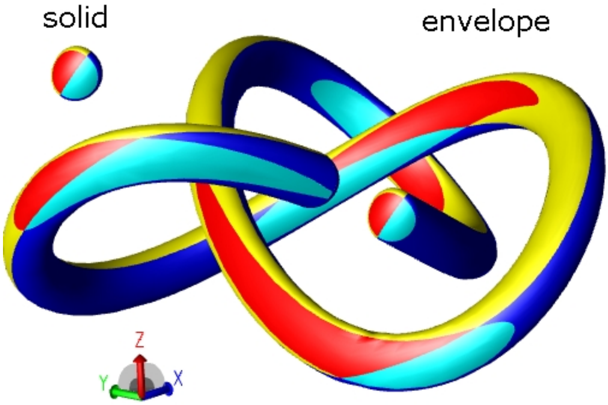
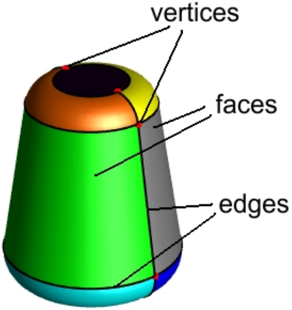


Figure: A solid swept along a trefoil knot.

Question: What is the meaning of **compute the swept volume** $\mathcal{V}$?

Ans: Input solid and output swept volume specified in **boundary representation** format.

The boundary representation (brep)



- **Geometric data:** Parametric definitions of faces, edges and vertices.
- **Topological data:** Orientation of faces and edges. Ajdacency relations amongst geometric entities.

Issues involved in brep computation

- When introducing a new surface type in a CAD kernel
  - Parametrization: Local aspects
  - Body check: Local and Global aspects
  - Topology: Global aspects
  - G1-discontinuity in input: Local and Global aspects
- Parametrization: **Funnel**
- Body check: **Self-intersection, Trim curves**.
- Topology: **Local homeomorphism** between input solid and envelope.
- G1-discontinuity: Sharp **edges** and **vertices** generate **faces** and **edges** respectively.

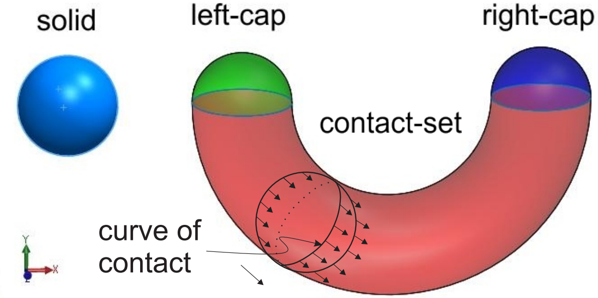
Parametrization

The envelope condition

- **Trajectory**
  $h : I \to (SO(3), \mathbb{R}^3)$, $h(t) = (A(t), b(t))$.
- **Trajectory of a point** x under h
  $\gamma_x : I \to \mathbb{R}^3$, $\gamma_x(t) = A(t) \cdot x + b(t)$.
- Define $g : \partial M \times I \to \mathbb{R}$ as $g(x, t) = \langle A(t) \cdot N(x), \gamma_x'(t) \rangle$.
- **Curve of contact** at t
  $C(t) = \{\gamma_x(t) \in \partial M(t) | g(x, t) = 0\}$.
- For $I = [t_0, t_1]$, the necessary condition for $\gamma_x(t)$ to belong to envelope $\mathcal{E}$:
  - If $t = t_0$ then $g(x, t) \leq 0$: **Left-cap**
  - If $t = t_1$ then $g(x, t) \geq 0$: **Right-cap**
  - If $t \in (t_0, t_1)$ then $g(x, t) = 0$: **Contact set**

The envelope condition

A point $\gamma_x(t)$ belongs to the contact-set only if the velocity $\gamma'_x(t)$ is tangent to ∂M at $\gamma_x(t)$.



Parametrization of envelope

- A suitable 2-dimensional sub-manifold of the parameter space of the sweep problem serves as the domain of parametrization of the envelope.
- The **procedural** approach leads to an accurate and efficient parametrization of the envelope.
- In this paradigm, the surface/curve definition is stored as numerical procedures, which, when invoked with the supplied parameter value, converge to the required point/derivative within specified tolerance.

Parametrization of envelope

- Parametric surface $S : \mathbb{R}^2 \to \mathbb{R}^3$, $S(D) = F \subseteq \partial M$.
- Define $f : D \times I \to \mathbb{R}$ as $f(u, v, t) = g(S(u, v), t)$
- **Funnel**: $\mathcal{F}^F = \{(u, v, t) \in D \times I | f(u, v, t) = 0\}$.
- **Parametrization map**: $\sigma^F : \mathcal{F}^F \to C^F$,
  $\sigma(u, v, t) = A(t) \cdot S(u, v) + b(t)$.

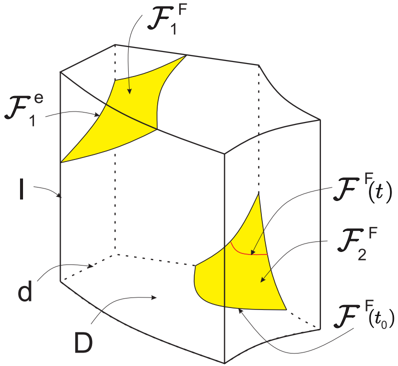


Figure: In this example, the funnel has two components, shaded in yellow.

Parametrization of envelope

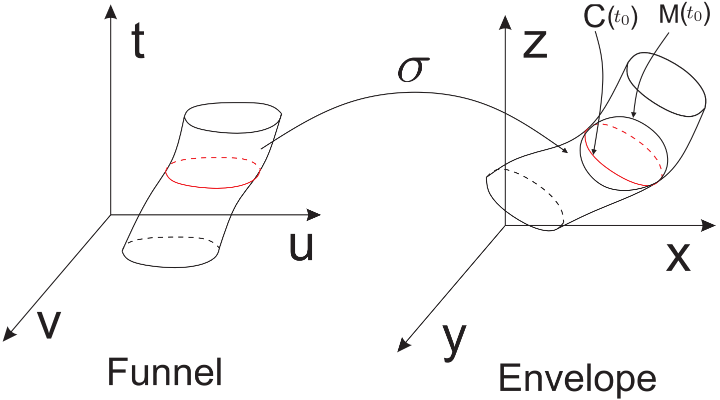
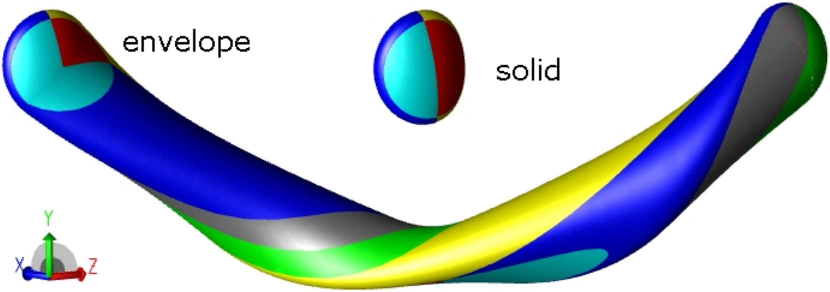


Figure: σ maps funnel ($\mathcal{F}$) in param. space to envelope ($\mathcal{E}$) in object space

Self-intersections

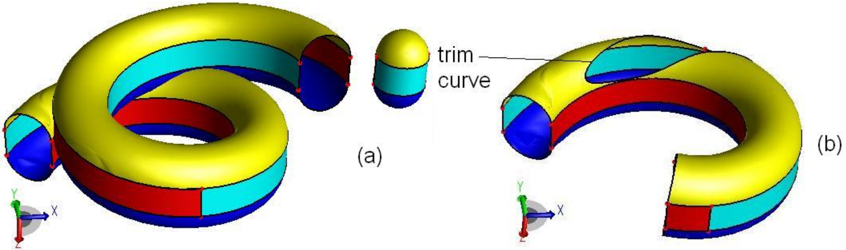
Simple sweeps

In simple sweeps, no trimming of the contact set is required to obtain the envelope.



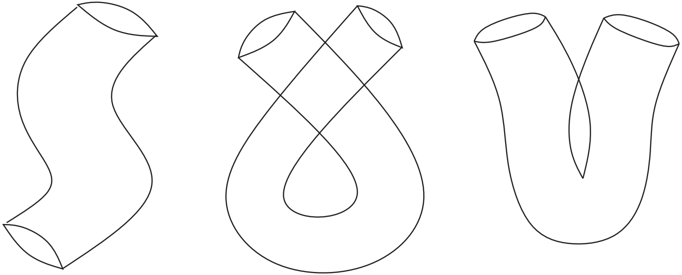
Not all sweeps are simple

- **Trim set** $T := \{x \in C | \exists t \in I, x \in M^o(t)\}$.
- **p-trim set** $pT := \sigma^{-1}(T) \cap \mathcal{F}$.
- **Trim curve** ∂T: boundary of $\overline{T}$.
- **p-trim curve**: ∂pT: boundary of $\overline{pT}$.



Self-intersections

A novel classification of sweeps into **simple**, **decomposable** and **non-decomposable** based on the complexity of trim curves.



(a) (b) (c)

Figure: (a)Simple sweep (b)Decomposable sweep (c)Non-decomposable sweep

A geometric invariant on $\mathcal{F}$

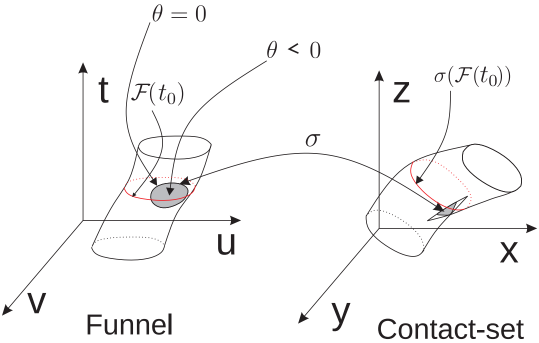
- For $p \in \mathcal{F}$, $\{\sigma_u(p), \sigma_v(p), \sigma_t(p)\}$ are **linearly dependent**.
- Let $\sigma_t(p) = n(p).\sigma_u(p) + m(p).\sigma_v(p)$, n and m continuous on $\mathcal{F}$.
- Define $\theta : \mathcal{F} \to \mathbb{R}$,

$$\theta(p) = n(p) \cdot f_u(p) + m(p) \cdot f_v(p) - f_t(p)$$

- If for all $p \in \mathcal{F}$, $\theta(p) > 0$, then the sweep is **decomposable**. If there exists $p \in \mathcal{F}$ such that $\theta(p) < 0$, then the sweep is **non-decomposable**.
- θ **invariant** of the parametrization of ∂M.
- Arises out of relation between two 2-frames on $\mathcal{T}_C$.
- Is a **non-singular** function.

A geometric invariant on $\mathcal{F}$

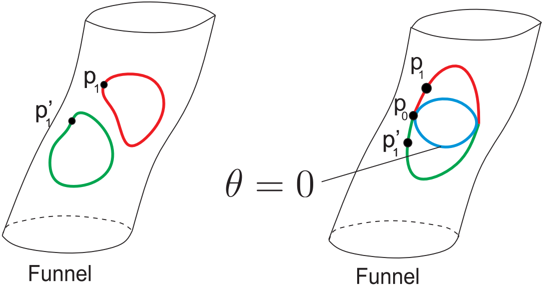
- θ partitions the $\mathcal{F}$ into (i) $\mathcal{F}^+ := \{p \in \mathcal{F} | \theta(p) > 0\}$, (ii) $\mathcal{F}^- := \{p \in \mathcal{F} | \theta(p) < 0\}$ and (iii) $\mathcal{F}^0 := \{p \in \mathcal{F} | \theta(p) = 0\}$.
- Define $C^+ := \sigma(\mathcal{F}^+)$, $C^- := \sigma(\mathcal{F}^-)$ and $C^0 := \sigma(\mathcal{F}^0)$.



- $C^- \subset T$.
- C^0: The set of points where $dim(\mathcal{T}_C) < 2$.

Trimming non-decomposable sweeps

The trim curve meets the zero locus of the invariant θ (shown in blue) in a non-decomposable sweep.



(a) Decomposable sweep (b) Non-decomposable sweep

Trimming non-decomposable sweeps

A **geometric invariant** θ leads to efficient classification of sweeps and aids in locating the trim curves in non-decomposable sweeps.

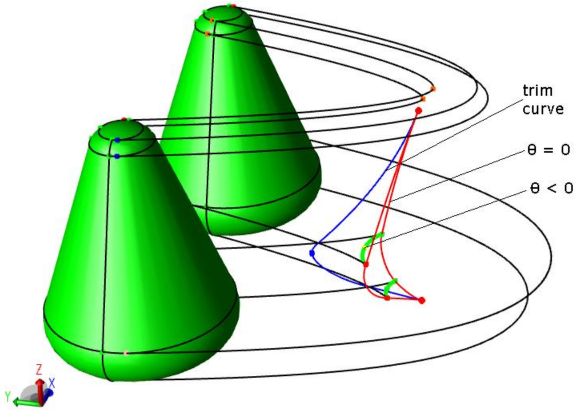


Figure: A cone being swept along a parabola. The trim curve, shown in blue, meets the zero locus of an invariant function θ, shown in red.

Topology

The natural correspondence between $\mathcal{E}$ and ∂M

- Correspondence $\pi : \mathcal{E} \to \partial M$, $\pi(y) = x$ such that $y = A(t) \cdot x + b(t)$ for some $t \in I$, i.e., y is a translate of x.
- Thanks to π, we lift the topological data of ∂M to that of $\mathcal{E}$.

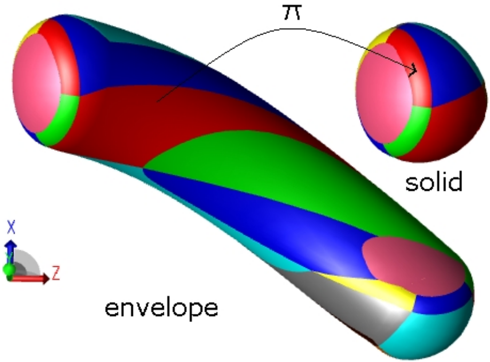
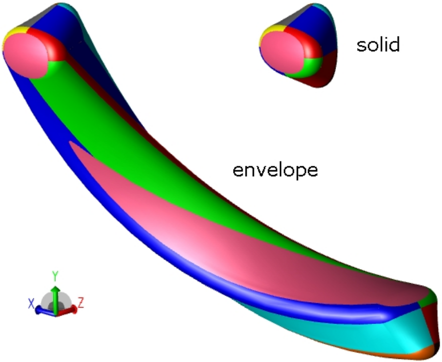


Figure: The points y and $\pi(y)$ are shown in same color.

Adjacency relations

The local homeomorphism $\pi : \mathcal{E} \to \partial M$ respects **adjacency relations** amongst faces, edges and vertices.



While the global brep structures of ∂M and $\mathcal{E}$ may be very different, locally they are **very** similar.

Orientation

The map $\pi : \mathcal{E} \to \partial M$ is orientation preserving if $-f_t > 0$ and reversing if $-f_t < 0$.

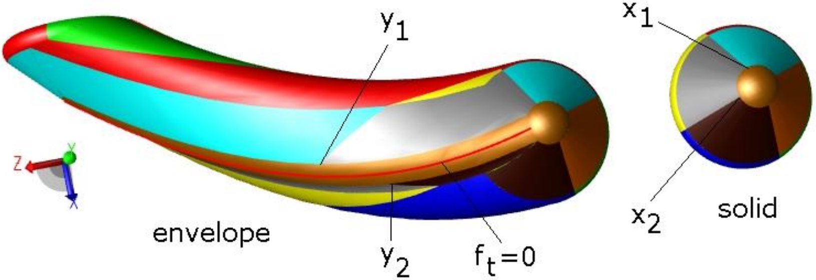
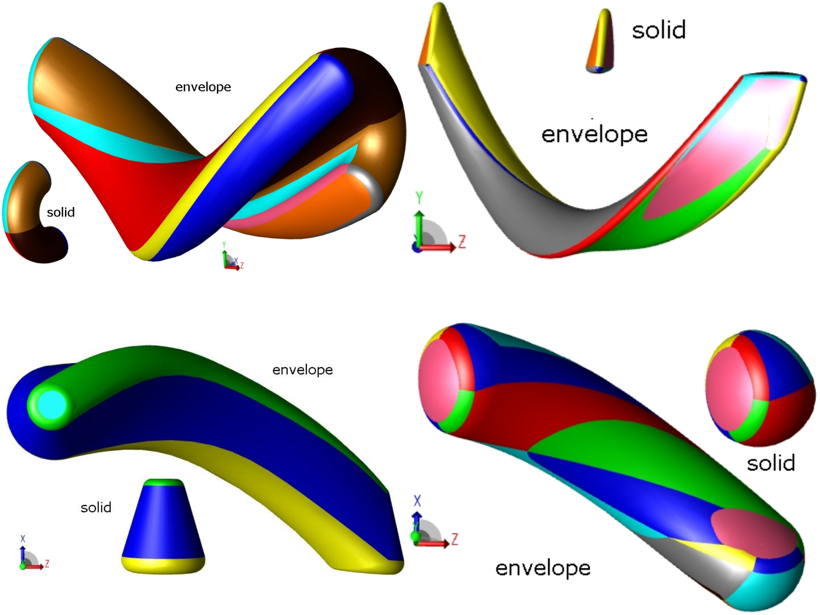


Figure: Here $\pi(y_i) = x_i$. The map π is orientation preserving at y_2 and reversing at y_1. The curve $f_t = 0$ is shown in red.

Algorithm 1 Solid sweep

```
for all faces F in ∂M do
    for all co-edges e in ∂F do
        for all vertices z in ∂e do
            Compute vertices C^z generated by z
        end for
        Compute co-edges C^e generated by e
        Orient co-edges C^e
    end for
    Compute C^F(t_0) and C^F(t_1)
    Compute loops bounding faces C^F generated by F
    Compute faces C^F generated by F
    Orient faces C^F
end for
for all F_i, F_j adjacent in ∂M do
    Compute adjacencies between faces in C^{F_i} and C^{F_j}
end for
```

Examples from a pilot implementation over **ACIS** kernel



Incorporating sharp features

A G1-discontinuous solid

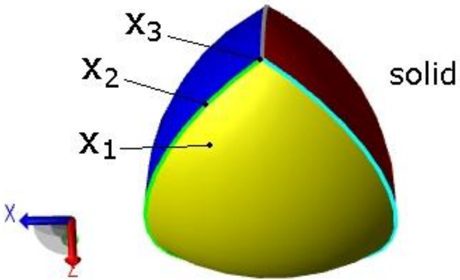
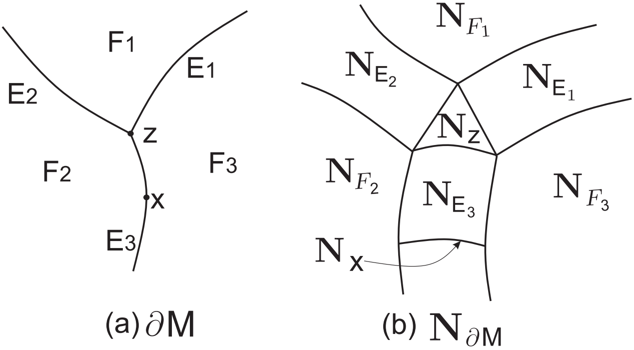


Figure: The points x_1, x_2 and x_3 belong to a smooth face, a sharp edge and a sharp vertex respectively.

There exists a **cone of unit normals** at each sharp point on the input solid.



Calculus of cones

- A sharp **edge** generated a set of *faces* on the envelope.
- Such faces are free of **local self-intersections**.
- A sharp **vertex** generates a set of **edges** on the envelope.

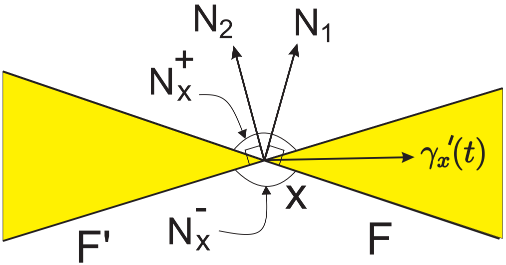


Figure: The point x is on the envelope only if the velocity lies in the region shaded in yellow.

Adjacency relations

The envelope has **degenerate** vertices but no **sharp** vertices.

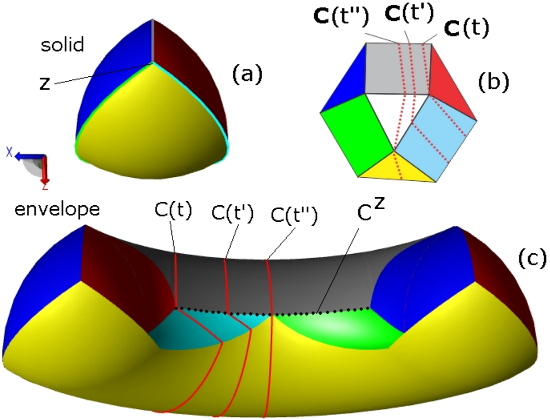
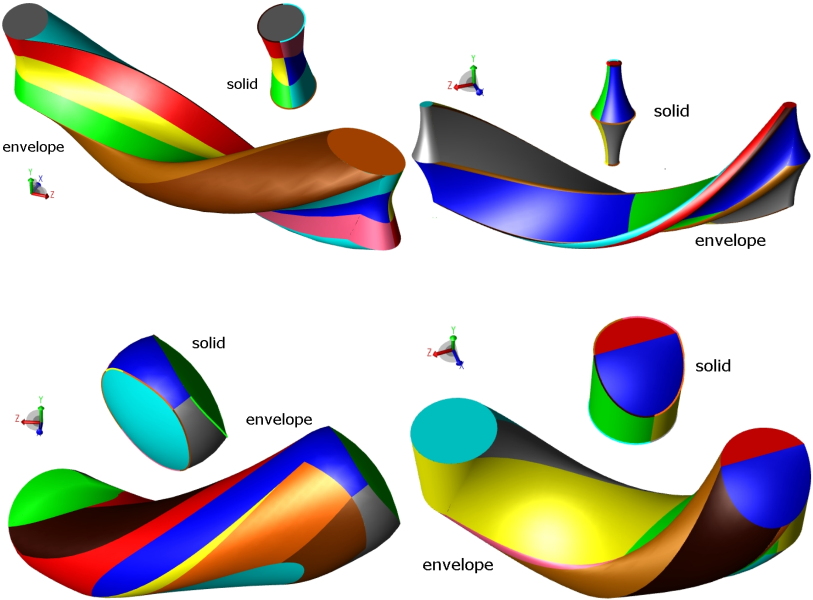
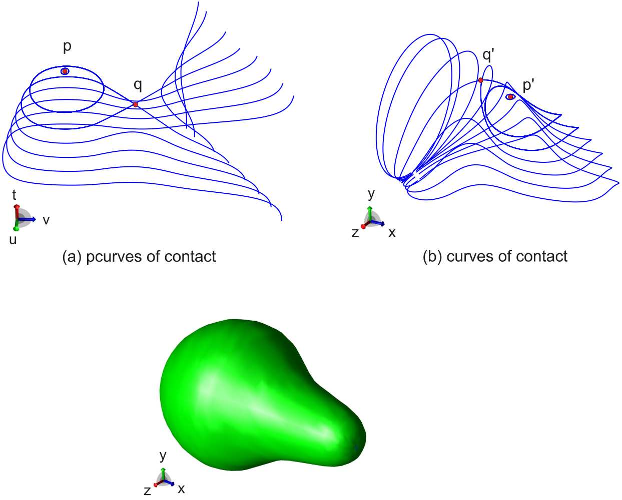


Figure: The edge C^Z generated by the sharp vertex $Z \subset \partial M$ is shown as a dotted curve in black on the envelope

Examples from a pilot implementation over **ACIS** kernel



Further: Non-simple curve of contact



(a) pcurves of contact

(b) curves of contact

(c) solid

Figure: A round-bottom flask undergoing curvilinear motion along an arc in *xy*-plane.

Summary of contribution

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- A complete and robust computational framework for solid sweep
- Accurate and fast parametrization via the *funnel* and *procedural* approach.
- A novel classification of sweeps based on complexity of trim curves.
- A geometric invariant which aids efficient classification and location of trim curves.
- Understanding of a brep structure induced on the envelope by the input solid.
- Handling G1 discontinuities in input solid.
- A pilot implementation over the ACIS solid modeling kernel.

References

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Thank You