Design and Analysis of Algorithms CS218M

Dynamic Programming

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Maximum-Weight Non Overlapping Set of Intervals

Problem Given a set of intervals $I_1, \ldots, I_n$ where each interval I_i has a start time $s(I_i)$, end time $f(I_i)$ and a weight $w(I_i)$ aim is to find a subset S of the intervals such that no two intervals in S overlap and the sum of weights of interval $\Sigma_{I_i \in S} \; w(i)$ is maximum.

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Optimal Recursive Substructure

Let $I \in S$. Then

$$MWNOSI(S) \ = \ Max(\ MWNOSI(S - \{I\}),\ MWNOSI(S') + w(I))$$

where S' is the set of intervals from S which do not overlap with I.

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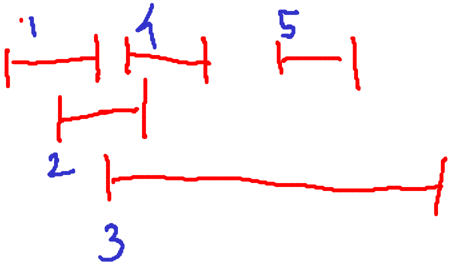

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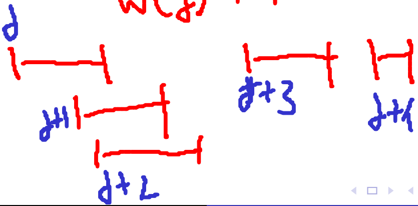
$$MWNOSI(j) = \max\left(MWNOSI(j+1),\; w(j) + MWNOSI(k)\right)$$



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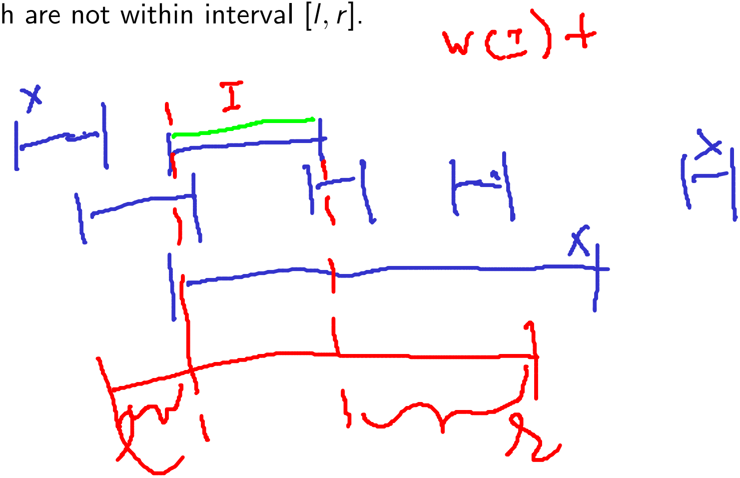
- Base case? $m[n+1] = 0$
- What is the structure of memoization table T. $m[1..n+1]$
- Complexity with memoization? $\Theta(1)$

$\Theta(n)$

Solution 2: Adding Extra Parameters

Let $EMWNOSI(I_1, \ldots, I_n;\ l;\ r)$ denote max weight subset of non-overlapping intervals from the list after removing intervals which are not within interval $[l, r]$.



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Base case:

Memoization Table m? dimensions?

Order of filling m?

Complexity with Memoization? $\Theta(n^3)$

Knapsack

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- Select items in decreasing order of value (most expensive first). Value $(3,2,2)$ with weight $(W/2+1,\ W/2,\ W/2)$.
- No known Greedy rule gives optimal solution.

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$i \leftarrow 0..n$

$0 \leq w \leq W$

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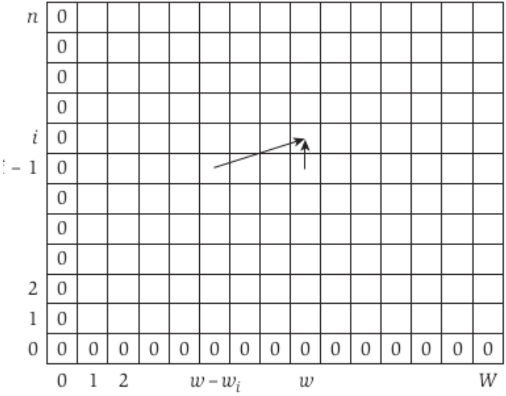
- Base cases: $opt(0, w) = 0$.
- Dimensions of memoization table m?
- Complexity? $O(n \cdot W)$
  Pseudo-polynomial – proportional to value of constant occuring in input.

Knapsack Bottom Up Schedule



Detecting Subsequence

Problem Given sequences (arrays) $X = \langle x_1, x_2, \ldots, x_m \rangle$ and sequence $Z = \langle z_1, \ldots, z_k \rangle$, determine whether Z is a subsequence of X, that is there exists a sequence of indices $\langle i_1, \ldots, i_k$ such that $Z_j = X_{i_j}$.

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Solution: Greedy Algorithm of Complexity $\Theta(m)$

Longest Common Subsequence (LCS)

Problem Given sequences (arrays) $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n$, determine the longest length sequence $Z = \langle z_1, \ldots, z_k \rangle$ which is a subsequence of both X and Y.

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Brute Force Solution

Systematically generate all subsequences Z of Y. For each check if Z is a subseqeunce of X. Also remember the maximum of the length of "yes" subsequences examined so far.

Complexity?

Optimal Substructure

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- If $x_m = y_n$ then $z_k = x_m = y_n$ and Z_{k-l} is LCS of X_{m-1}, Y_{n-1}.
- If $x_m \neq y_n$ and $z_k \neq x_m$ then

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Designing the DP algorithm

- Memoization table $c[0..m, 0..n]$.

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- Base case?, Order of computing $c[i,j]$?
- Complexity? $\Theta(m \cdot n)$

Bottom UP Procedure

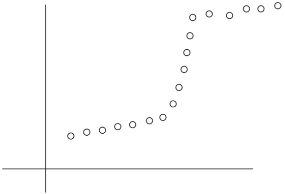
```
LCS-LENGTH(X, Y)
m = X.length
n = Y.length
let b[1..m, 1..n] and c[0..m, 0..n] be new tables
for i = 1 to m
    c[i, 0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i − 1, j − 1] + 1
            b[i, j] = "↖"
        elseif c[i − 1, j] ≥ c[i, j − 1]
            c[i, j] = c[i − 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j − 1]
            b[i, j] = "←"
return c and b
```

Constructing the Solution

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

Segmented Least Square

Problem [KT6.3] Given a set of n points $(x_1, y_1), \ldots, (x_n, y_n)$ in x, y-plane in order $x_1 < x_2 < \ldots < x_n$, find a small set of line segments such that the soln. gives the least error squared.



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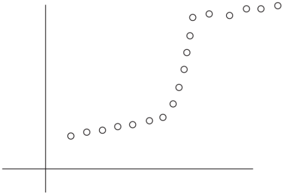

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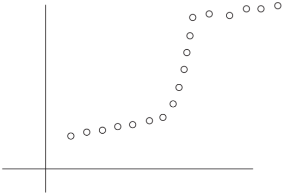

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Formulating the Problem

- if $p_i, p_{i+1}, \ldots, p_j$ belongs to a line segment then $e_{i,j}$ denotes the squared error from these points after fitting the best line through them.
- Trade off between reducing error and reducing number of line segments.
- Each line segment incurs a cost of C.

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Optimal Substructure

For the subproblem $p_1, \ldots, p_j$

$$opt(j) \;=\; min_{1 \leq i \leq j} \;\; e_{i,j} + C + opt(i-1)$$