Design and Analysis of Algorithms CS218M

NP Completeness

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Complexity Class $\mathbb{P}$: Polynomial time Solvable Problems

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P is class of problems with algorithm A_p and constant c s.t. time complexity of A_p $O(n^c)$ for input of size n.

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- Example:
  $PRIMES = \{x \in \{0,1\}^* \mid x \text{ represents a prime in binary}\}$.

$$Q = \{x \in \Sigma^* \mid x \text{ is yes instance of } Q\}$$

$$\overline{Q} = \{x \in \Sigma^* \mid x \text{ is no instance of } Q\}$$

$$= \Sigma^* - Q$$

Given class of Problem $\mathcal{C}$

$$\text{Co-}\mathcal{C} = \{\overline{Q} \mid Q \in \mathcal{C}\}$$

Examples

- *SAT* set of all strings representing satisfiable boolean formulas.

$$((x_1 \wedge x_2) \vee \neg x_3)$$

Facts

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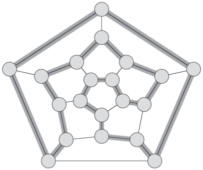
- *HAMILTON* is collection of all graphs which have a Hamiltonian cycle.



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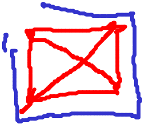

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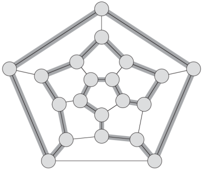

- *EULER* is collectionof all graphs which have a Euler cycle.
- *PATH* all graphs having a path from specified s to t of length at most k.

atmost k edges on path

$\langle G, s, t, k \rangle$

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- $EULER \in \mathbb{P}$ (why?)
- $PATH \in \mathbb{P}$ (why?)

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- Answer: Yes
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- Examples:
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- Above examples belong to an important class of problems called $\mathbb{NP}$ problems.

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- Example: For boolean formula
  $((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$
  certificate $x_1 = 0,\ x_2 = 0,\ x_3 = 1,\ x_4 = 1$

Definition of Class $\mathbb{NP}$

Definition

$Q \in \mathbb{NP}$ if and only if there exists a two input algorithm $A(X, Y) \in \mathbb{P}$ and a constant c such that

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Algorithms to Solve Q: Given such A and c, for any input X

- Nondeterministically guess Y. Output $A(X, Y)$.
- Systematically enumerate every Y and compute $A(X, Y)$. If any instance $A(X, Y)$ has answer "yes" then we answer "yes". Otherwise answer "no". The time of this Deterministic algorithm is $O(2^{Poly(|X|)})$.

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$\mathbb{P}$ versus $\mathbb{NP}$

- *SAT* $\in \mathbb{NP}$ (How?)
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Theorem

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$\mathbb{P} \subseteq \mathbb{NP}$

Natural Question: Is *SAT* $\in \mathbb{P}$

A Major Open Problem in CS [Cook71-Levin73]

Is $\mathbb{P} = \mathbb{NP}$?

ABOUT PROGRAMS PEOPLE MILLENNIUM PROBLEMS PUBLICATIONS EVE

The Millennium Prize Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) established seven *Prize Problems*. The Prizes were conceived to record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium; to elevate in the consciousness of the general public the fact that in mathematics, the frontier is still open and abounds in important unsolved problems; to emphasize the importance of working towards a solution of the deepest, most difficult problems; and to recognize achievement in mathematics of historical magnitude.

Clay Millenium Problems (List)

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

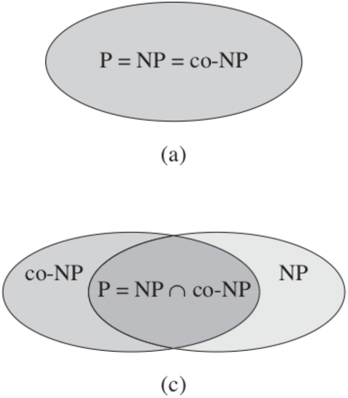
Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Potential Scenarios



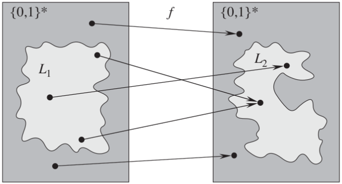
There is strong belief (but no proof!) that $\mathbb{P} \subset \mathbb{NP}$.

Polynomial Time Reduction

Let $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Delta^*$ be two decision problems. A function $f : \Sigma^* \to \Delta^*$ is called a polynomial time reduction from L_1 to L_2 (denoted $f : L_1 \leq_P L_2$) iff

- f is total.
- There exists a c such that $f(x)$ is computable in time $O(|x|^c)$ for all x.
- $x \in L_1$ iff $f(x) \in L_2$.

Notation $L_1 \leq_P L_2$ denotes $\exists f$ such that $f : L_1 \leq_P L_2$.

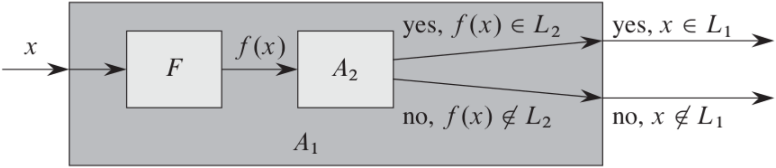


Using Poly Time Reduction

Theorem

If $f : L_1 \leq_P L_2$ and $L_2 \in \mathbb{P}$ then $L_1 \in \mathbb{P}$.

Proof:



Time taken for computing A_1 is polynomial in x (why?)

$\mathbb{NP}$-Complete Problems

A problem $L \subseteq \Sigma^*$ is called $\mathbb{NP}$-complete ($\mathbb{NPC}$) if

- $L \in \mathbb{NP}$.
- For every $L' \in \mathbb{NP}$, we have $L' \leq_P L$.
  (This shows that L is at least as hard as L'.)

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Theorem

If $L' \leq_P L$ and L' is $\mathbb{NPC}$ then L is $\mathbb{NP}$-hard. Additionally if $L \in \mathbb{NP}$ then L is $\mathbb{NPC}$.

We can prove a wide class of problems to be $\mathbb{NPC}$.

- (Initial Problem) Circuit Satisfaction Problem: Given a combinational circuit made out of AND, OR, NOT gates, decide whether any input makes the circuit output 1.
- (More problems by Reduction) Independat Set, Vertex Cover, k-Clique, set Packing, set Cover, sat, 3-sat, Hamiltonian Cycle, Travelling Salesman, 3-dimnesional-matching, graph-coloring, subset-sum