Design and Analysis of Algorithms CS218M

NP Complete Problems

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$\mathbb{NP}$-Complete Problems

A problem $L \subseteq \Sigma^*$ is called $\mathbb{NP}$-complete (denoted $\mathbb{NPC}$) if

- $L \in \mathbb{NP}$.
- For every $L' \in \mathbb{NP}$, we have $L' \leq_P L$.
  (This shows that L is at least as hard as L'.)

If only second condition is satisfied we say that L is $\mathbb{NP}$-hard.

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Theorem

If $L' \leq_P L$ and L' is $\mathbb{NPC}$ then L is $\mathbb{NP}$-hard. Additionally if $L \in \mathbb{NP}$ then L is $\mathbb{NPC}$.

If $L' \le_p L$ and $L' \in NPC$

then $L \in NP\text{-hard}$

Proof For any $L'' \in NP$

$L'' \le_p L'$ (why)

$L' \le_p L$

$\Rightarrow L'' \le_p L$

$3CNF\_SAT$ is $\mathbb{NP}$-Complete

Formula in Conjunctive Normal Form (CNF)

- Example $(\neg x_v \vee x_u \vee x_z) \wedge (\neg x_v \vee x_w \vee \neg x_z) \wedge (x_v \vee \neg x_u \vee \neg x_w)$

$3CNF\_SAT$ is $\mathbb{NP}$-Complete

- Example $(\neg x_v \vee x_u \vee x_z) \wedge (\neg x_v \vee x_w \vee \neg x_z) \wedge (x_v \vee \neg x_u \vee \neg x_w)$
- A boolean formula ϕ in the form $C_1 \wedge C_2 \wedge \ldots \wedge C_m$ where each clause C_i has the form $(l_1^i \vee l_2^i \vee l_3^i)$ where literal l is x or $\neg x$ for a propositional letter x is called $3CNF$ formula.

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- $3CNF\_SAT$ is the collection of satisfiable $3CNF$ formulas. It is easy to see that $3CNF\_SAT \in \mathbb{NP}$ (why?)

4-DNF-SAT
2-CNF-SAT

DNF Formula

$(x_1 \land \neg x_3 \land x_5) \lor$

$(x_2 \land x_3 \land \neg x_4) \lor$

$\vdots$

$3CNF\_SAT$ is $\mathbb{NP}$-Complete

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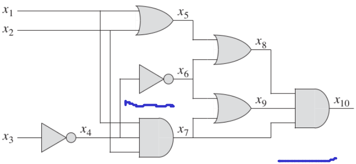
- We show that $CIRCUIT\_SAT \ \leq_P \ 3CNF\_SAT$.

Reduction: We trasform a circuit C to formula $\phi(C)$

$(x_6 \leftrightarrow \neg x_4)$



$$
\begin{aligned}
\phi = x_{10} & \wedge (x_4 \leftrightarrow \neg x_3) \\
& \wedge (x_5 \leftrightarrow (x_1 \vee x_2)) \\
& \wedge (x_6 \leftrightarrow \neg x_4) \\
& \wedge (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\
& \wedge (x_8 \leftrightarrow (x_5 \vee x_6)) \\
& \wedge (x_9 \leftrightarrow (x_6 \vee x_7)) \\
& \wedge (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9)) .
\end{aligned}
$$

$(x_{10} \leftrightarrow x_7 \wedge Y_2)$

$(Y_2 \leftrightarrow x_8 \wedge x_9$

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- Transform circuit C to conjunction of clauses as in previous slide.
- A multi-input AND is replaced by a cascade of 2-input AND with extra variables (Also OR.)

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  Clause $x_v \leftrightarrow \neg x_u$ gives rise to equivalent set of clauses
  $(x_v \vee x_u) \wedge (\neg x_v \vee \neg x_u)$.

$x_v \wedge \neg x_u \vee \neg x_v \wedge x_u$

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- Each two literal clause is converted to a set of 3-literal clauses.
  E.g. $(x \vee y)$ is equivalent to $(x \vee y \vee p) \wedge (x \vee y \vee \neg p)$.

Rectified 3CNF Formula

1 $x \wedge \ldots$

$(x_1 \vee \neg x_2) \wedge$

$(x_1 \vee \neg x_2 \vee p) \wedge (x_1 \vee \neg x_2 \vee \neg p)$

2 $(x_1 \vee \cancel{x_1} \vee y)$ ✗

$(\cancel{x_1 \vee \neg x_1} \vee y)$ 1

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Theorem

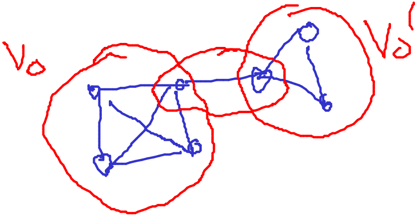
C is satisfiable iff $\phi(C)$ is satisfiable. Also $|\phi(C)|$ is linear in $|C|$. Hence, $CIRCUIT\_SAT \leq_P 3CNF\_SAT$.

CLIQUE

undirected

Given a graph G a subset $V_0 \subseteq V$ is a clique if for every distinct $u, v \in V_0$ we have $(u, v) \in E$.

$$CLIQUE = \{\langle G, k \rangle \mid G \text{ has a clique of size } k\}$$



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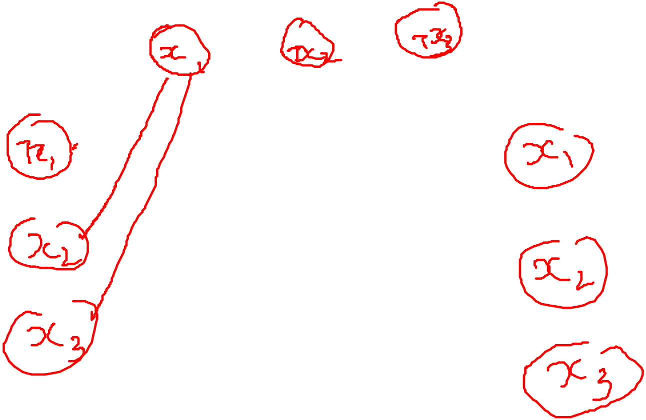
$CLIQUE \in \mathbb{NPC}$

- $CLIQUE \in \mathbb{NP}$ (How?)
- $3CNF\_SAT \leq_P CLIQUE$.

Reduction $3CNF\_SAT \leq_P CLIQUE$

$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$



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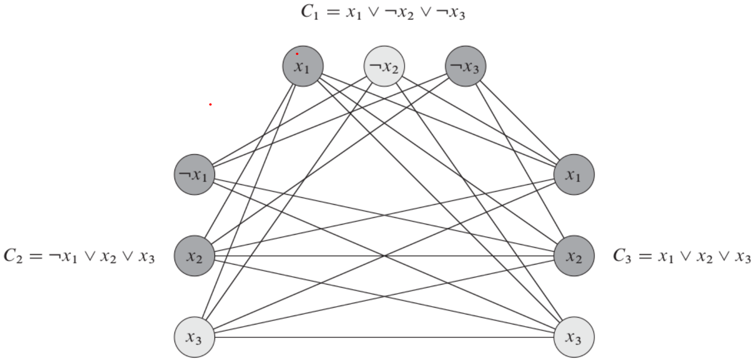

Figure 34.14 The graph G derived from the 3-CNF formula $\phi = C_1 \wedge C_2 \wedge C_3$, where $C_1 = (x_1 \vee \neg x_2 \vee \neg x_3)$, $C_2 = (\neg x_1 \vee x_2 \vee x_3)$, and $C_3 = (x_1 \vee x_2 \vee x_3)$, in reducing 3-CNF-SAT to CLIQUE. A satisfying assignment of the formula has $x_2 = 0$, $x_3 = 1$, and x_1 either 0 or 1. This assignment satisfies C_1 with $\neg x_2$, and it satisfies C_2 and C_3 with x_3, corresponding to the clique with lightly shaded vertices.

Clique, Independant Set, Vertex Cover

Given a graph G a subset $V_0 \subseteq V$ is

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- vertex cover iff for every edge $(u, v) \in E$ we have $(u \in V_0 \vee v \in V_0)$.
  $VERTEX\_COVER =$
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We explore reductions between these decision problems.

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Theorem

Let $G = (V, E)$ be a given graph and let the complement graph $G' = (V, \overline{E})$ where $\overline{E} = V^2 - E$. Then, For any $V_0 \subseteq V$, we have V_0 is a clique in G iff V_0 is an independant set in G'.

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- SELF STUDY: There is a nice generalization of independant set in graph to a problem called set packing. (See KT 8.1)

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Given undirected graph $G = (V, E)$ has a k coloring $f : V \to \{1, 2, \ldots, k\}$ if $(u, v) \in E \Rightarrow f(u) \neq f(v)$.

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Graph Coloring

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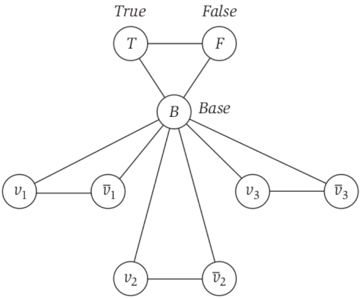
- Whether a graph is 3-colorable is in $3COLOAR \in \mathbb{NP}$.
- $3CNF\_SAT \leq_P 3COLOR$

Proof Idea

- For each variable x_i we have nodes v_i and $\overline{v}_i$.
- Encoding valuation by 3-coloring.



Proof Idea (2)

Enforcing clause $(x_1 \vee \neg x_2 \vee x_3)$.

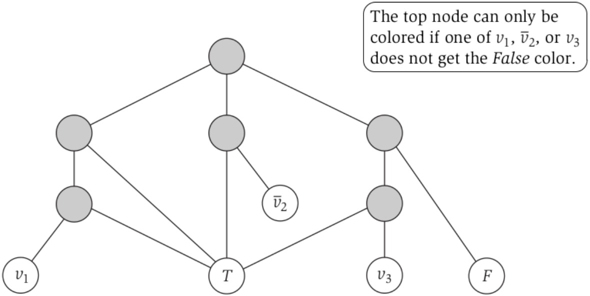


Figure 8.12 Attaching a subgraph to represent the clause $x_1 \vee \overline{x}_2 \vee x_3$.