Design and Analysis of Algorithms CS218M

NP Complete Problems

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$\mathbb{NP}$-Complete Problems

A problem $L \subseteq \Sigma^*$ is called $\mathbb{NP}$-complete (denoted $\mathbb{NPC}$) if

- $L \in \mathbb{NP}$.
- For every $L' \in \mathbb{NP}$, we have $L' \leq_P L$.
  (This shows that L is at least as hard as L'.)

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If $L' \leq_P L$ and L' is $\mathbb{NPC}$ then L is $\mathbb{NP}$-hard. Additionally if $L \in \mathbb{NP}$ then L is $\mathbb{NPC}$.

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- A boolean formula ϕ in the form $C_1 \wedge C_2 \wedge \ldots \wedge C_m$ where each clause C_i has the form $(l^i_1 \vee l^i_2 \vee l^i_3)$ where literal l is x or $\neg x$ for a propositional letter x is called $3CNF$ formula.

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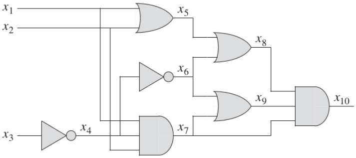
- We show that $CIRCUIT\_SAT \ \leq_P \ 3CNF\_SAT$.

Reduction: We trasform a circuit C to formula $\phi(C)$



$$\begin{aligned}\phi = x_{10} & \wedge (x_4 \leftrightarrow \neg x_3) \\ & \wedge (x_5 \leftrightarrow (x_1 \vee x_2)) \\ & \wedge (x_6 \leftrightarrow \neg x_4) \\ & \wedge (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\ & \wedge (x_8 \leftrightarrow (x_5 \vee x_6)) \\ & \wedge (x_9 \leftrightarrow (x_6 \vee x_7)) \\ & \wedge (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9)) .\end{aligned}$$

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Theorem

C is satisfiable iff $\phi(C)$ is satisfiable. Also $|\phi(C)|$ is linear in $|C|$. Hence, $CIRCUIT\_SAT \leq_P 3CNF\_SAT$.

CLIQUE

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- $3CNF\_SAT \leq_P CLIQUE$.

Reduction $3CNF\_SAT \leq_P CLIQUE$

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

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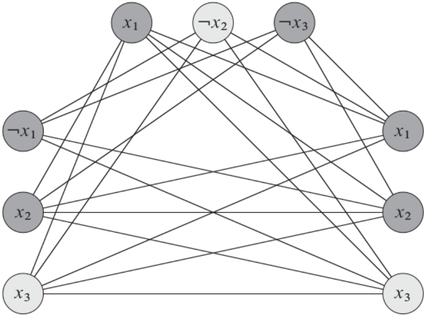
$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$

$C_2 = \neg x_1 \vee x_2 \vee x_3$



$C_3 = x_1 \vee x_2 \vee x_3$

Figure 34.14 The graph G derived from the 3-CNF formula $\phi = C_1 \wedge C_2 \wedge C_3$, where $C_1 = (x_1 \vee \neg x_2 \vee \neg x_3)$, $C_2 = (\neg x_1 \vee x_2 \vee x_3)$, and $C_3 = (x_1 \vee x_2 \vee x_3)$, in reducing 3-CNF-SAT to CLIQUE. A satisfying assignment of the formula has $x_2 = 0$, $x_3 = 1$, and x_1 either 0 or 1. This assignment satisfies C_1 with $\neg x_2$, and it satisfies C_2 and C_3 with x_3, corresponding to the clique with lightly shaded vertices.

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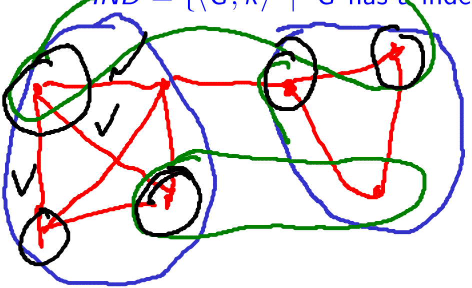

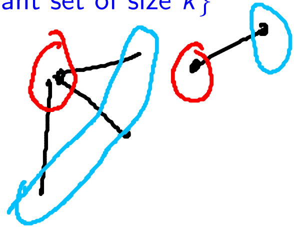

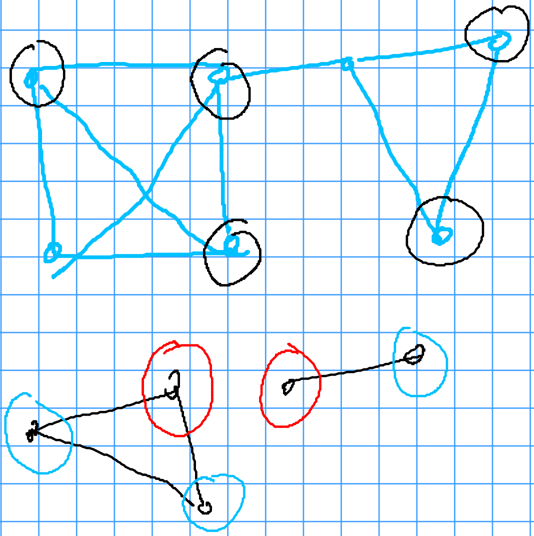

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- vertex cover iff for every edge $(u, v) \in E$ we have $(u \in V_0 \vee v \in V_0)$.
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We explore reductions between these decision problems.

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Let $G = (V, E)$ be a given graph and let the complement graph $G' = (V, \overline{E})$ where $\overline{E} = V^2 - E$. Then, For any $V_0 \subseteq V$, we have V_0 is a clique in G iff V_0 is an independant set in G'.

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- SELF STUDY: There is a nice generalization of independant set in graph to a problem called set packing. (See KT 8.1)

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Graph Coloring

Given undirected graph $G = (V, E)$ has a k coloring $f : V \to \{1, 2, \ldots, k\}$ if $(u, v) \in E \Rightarrow f(u) \neq f(v)$.

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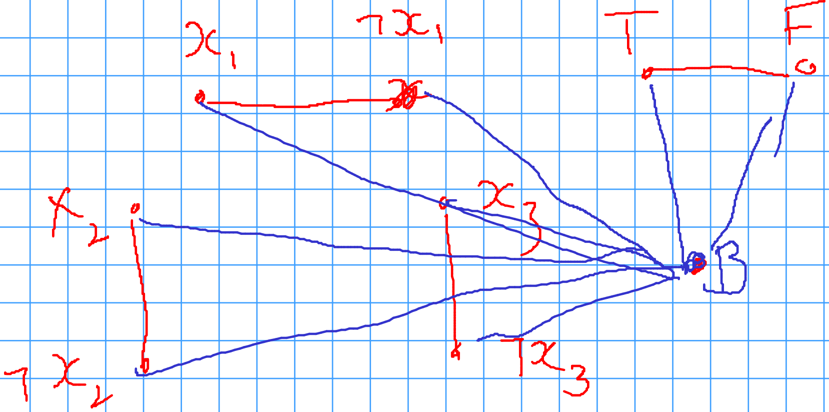

x_1
$\neg x_1$
T
F
x_2
x_3
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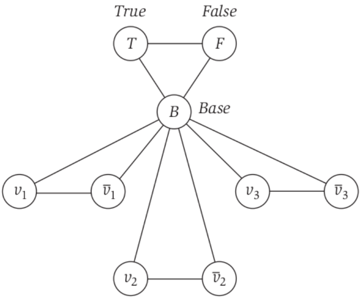
- $3CNF\_SAT \leq_P 3COLOR$

Proof Idea

- For each variable x_i we have nodes v_i and $\overline{v}_i$.
- Encoding valuation by 3-coloring.



$(x_1 \vee x_2 \vee \neg x_3)$

Proof Idea (2)

Enforcing clause $(x_1 \vee \neg x_2 \vee x_3)$.

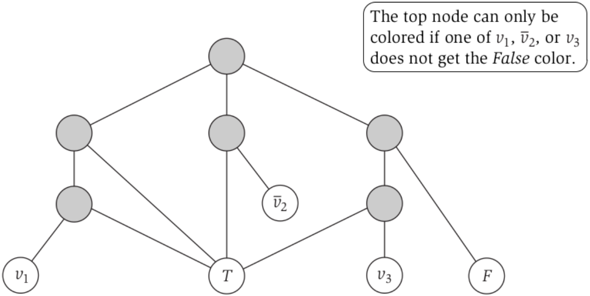


Figure 8.12 Attaching a subgraph to represent the clause $x_1 \vee \overline{x}_2 \vee x_3$.

Hamiltonian Cycle

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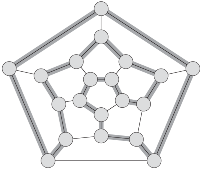

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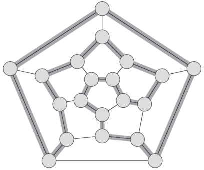

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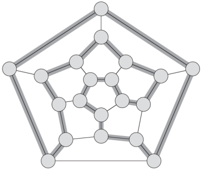

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Proof that $VERTEX\_COVER \leq_P HAM\_CYCLE$ is in book (CLRS 34.5.3). Students may read it out of interest.

Travelling Salesman Problem

Problem *TSP* Given a complete directed graph $G = (V, V \times V)$ with non-negative integer weights $c : V \times V \rightarrow Z_0$ does there exist a Hamiltonian cycle (called tour) whose weight is at most k?

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- G has a Hamiltonian cycle iff $(G', c, 0)$ has a tour of weight 0.

Subset Sum Problem

Problem Given a finite set of positive integers S and an integer $t > 0$ is there a subset $S' \subseteq S$ s.t. $(\Sigma_{i \in S'} \; i) \; = \; t$.

$S = \{1, 7, \overline{3}, 9, \overline{8}\}$

$t = 11$

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  Proof in CLRS 34.5.5 (Only for interested).

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- Consider the QBF formula $\exists x_1 \forall x_2 \exists x_3.\ \phi(x_1, x_2, x_3)$.