Design and Analysis of Algorithms CS218M

Network Flow Algorithms

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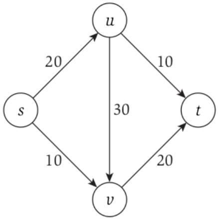
Optimal Network Flow Problem

A flow network is a directed graph $G = (V, E)$ where each edge (u, v) has a non-negative integer capacity $c(u, v) \geq 0$. The graph has source vertex s and sink vertext t.



- s has no incoming edges. t has no outgoing edges.
- For all internal nodes v there is a path $s \rightsquigarrow v \rightsquigarrow t$.

Flow

Given a flow network (G, c), a flow is $f : E \rightarrow Z_0$ such that

- (capacity) $0 \leq f(u, v) \leq c(u, v)$.
- (conservation) For all internal nodes v we have
  $$\Sigma_{e \text{ in to } v} f(e) = \Sigma_{e \text{ out of } v} f(e)$$

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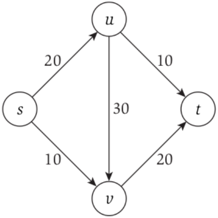

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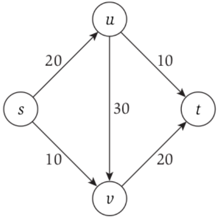

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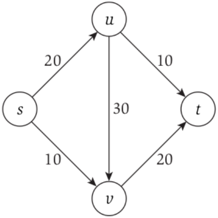

- Let $f^{out}(v) = \Sigma_{e \text{ out of } v} f(e)$ and $f^{in}(v) = \Sigma_{e \text{ in to } v} f(e)$.

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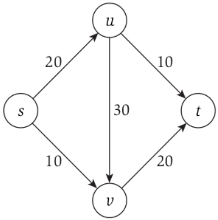
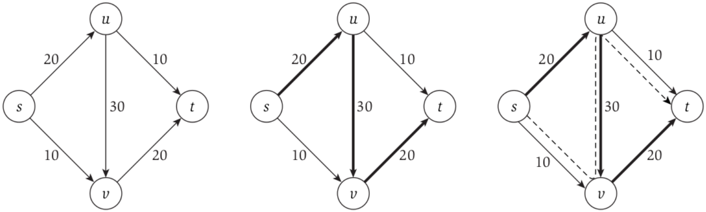


- Let $f^{out}(v) = \Sigma_{e \text{ out of } v} f(e)$ and $f^{in}(v) = \Sigma_{e \text{ in to } v} f(e)$.
- Define value of flow f as $f^{out}(s)$.

Finding Maximum Flow



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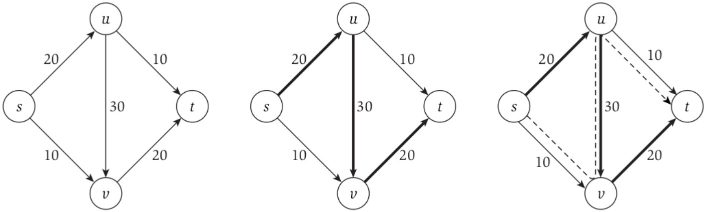

- Find a path $P = s \rightsquigarrow t$. Let $Bottleneck(P)$ be the smallest capacity on the path. Initial flow f has value $Bottleneck(P)$.

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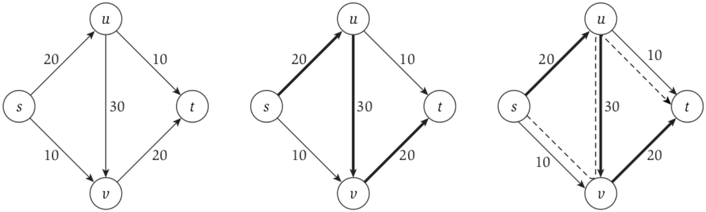
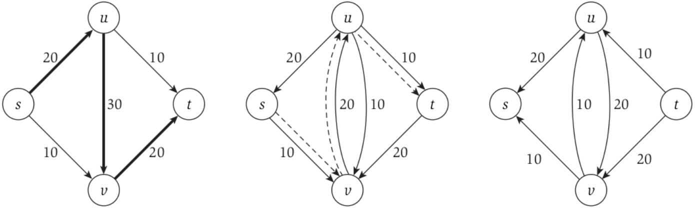


- Find a path $P = s \rightsquigarrow t$. Let $Bottleneck(P)$ be the smallest capacity on the path. Initial flow f has value $Bottleneck(P)$.
- Given current flow f, find augmenting path $P = s \rightsquigarrow t$. Check that it is feasible and push $Bottleneck(P)$ additional flow to get revised flow f'.

Residual Graph

Given flow f for flow network G, c, define Residual graph G_f.



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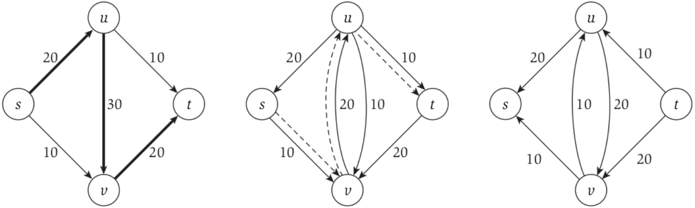

- Forward edges: Edges e with residual capacity $c(e) - f(e) > 0$.

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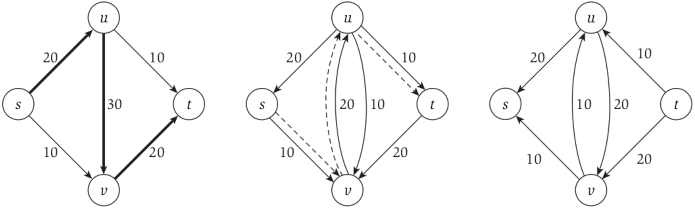
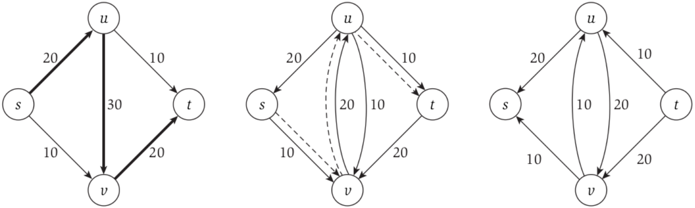


- Forward edges: Edges e with residual capacity $c(e) - f(e) > 0$.
- Backward edges: Reverse $\overline{e}$ of edges e with $f(e) > 0$ allowing reverse flow upto $f(e)$.

Augmenting the Flow



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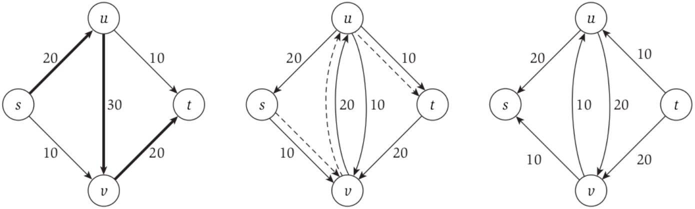

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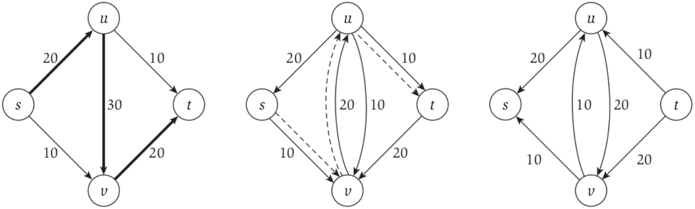

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- Augmented Flow f' is f modified as follows:
  - for each forward edge e on P, increase $f'(e) = f(e) + b$
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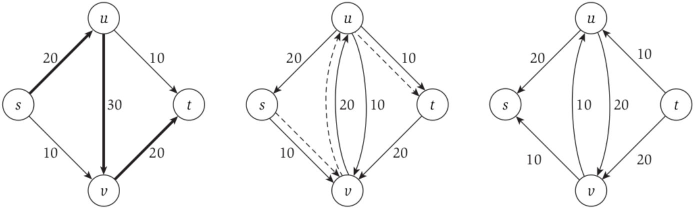

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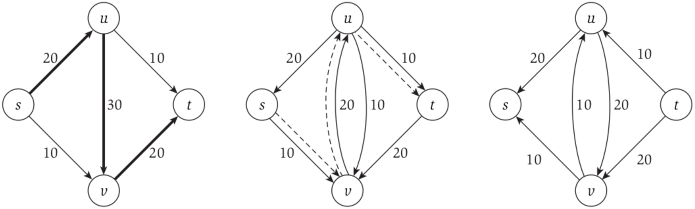

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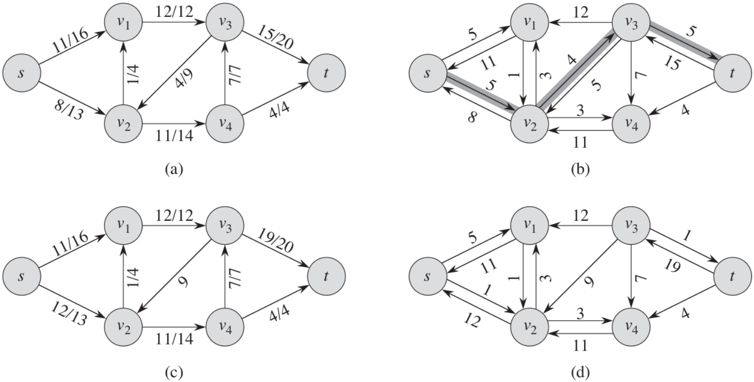
- Augmentation f' from f can be computed in time $O(E)$.

Ford-Fulkerson Algorithm (1956) for Max-Flow

```
Max-Flow
  Initially f(e) = 0 for all e in G
  While there is an s-t path in the residual graph G_f
    Let P be a simple s-t path in G_f
    f' = augment(f, P)
    Update f to be f'
    Update the residual graph G_f to be G_f'
  Endwhile
  Return f
```

Example: Ford Fulkerson



Max-flow Min-Cut Theorem

Given flow network G, c and a valid flow f,

- partition A, B of V is an s, t-cut if $s \in A$ and $t \in B$.

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Theorem

If $f*$ is the flow such that there is no $s \rightsquigarrow t$ path in G_f (i.e. $f*$ is returned by Ford-Fulkerson algorithm), then we can construct an s, t cut $A, B*$ such that $f* = c(A*, B*)$. Hence $f*$ is max flow and $A*, B*$ is min cut.*

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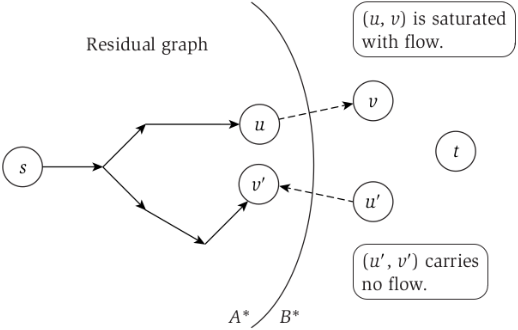
Construction: Let $A*$ be all nodes v s.t. $s \rightsquigarrow v$ in G_f. Let $B* = V - A*$.

Proof Idea



Residual graph
(u, v) is saturated with flow.
v
u
s
t
v′
u′
(u′, v′) carries no flow.
A*
B*

Integrality of Maximal Flow

If flow network G, c is such that $c(e)$ is non-negative integer for each e, then maximum flow $f*$ produced by Fork Fulkerson algorithm assigns integer flow value to each edge.

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At each iteration, the Ford-Fulkerson algorithm augments the flow with only integral value. Hence, flow value in each edge at each iteration is invariantly integral.

Maximal Matching in Bipartate Graph

$V = X \uplus Y$

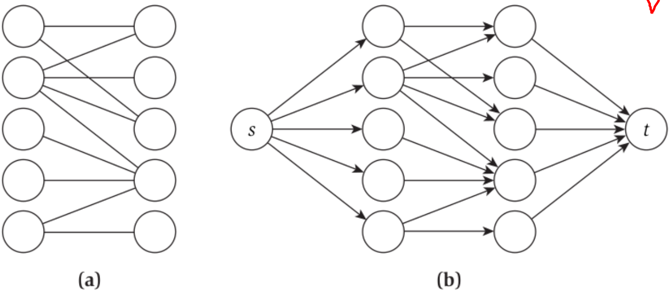

Figure 7.9 (a) A bipartite graph. (b) The corresponding flow network, with all capacities equal to 1.

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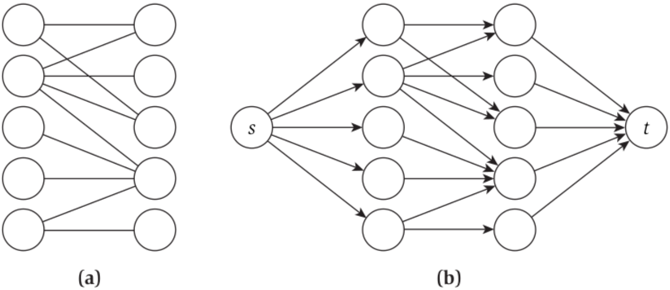

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- Bipartate graph $(X; Y, E)$. Figure (a).

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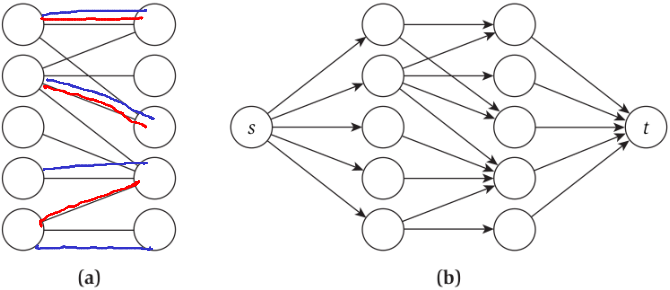

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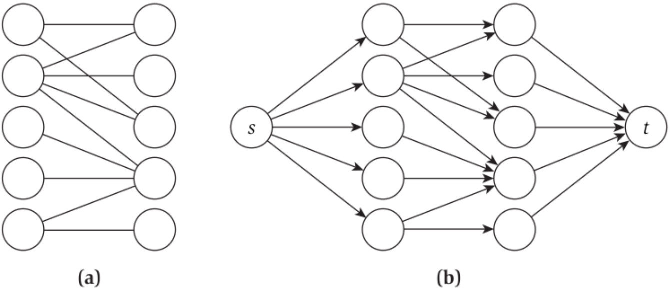

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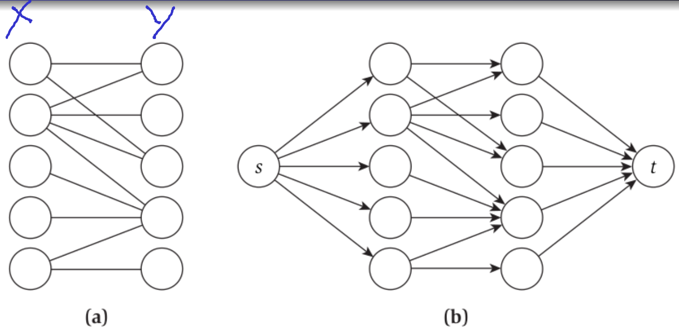

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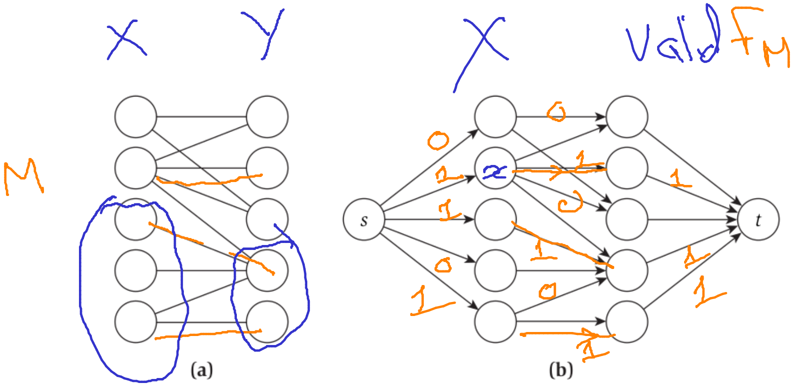
- Maximal Matching.
- Perfect Matching: Every $v \in X \cup Y$ occurs exactly once in M

Illustration



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Reduction to Flow Network

Given bipartate graph $(X; Y, E)$, we can construct a flow graph (G', c) with every edge e having capacity $c(e) = 1$. See figure.

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- *If M is a matching, then corresponding flow f_M obtained by assigning $f_M(e) = 1$ if $e \in M$ and $f_M(e) = 0$ otherwise is a valid integral flow of G', c.* $|M| = v(f_M)$

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1 $O(|V||E|)$

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Corollary

If $f*$ is a maximal integral flow then M_{f*} is a maximal matching.

Hall's Theorem

A bipartate graph $G = (X; Y, E)$ with $|X| = |Y| = n$

- has a perfect matching

  if and only iff

- for all $A \subseteq X$ we have $|A| \leq |E(A)|$.

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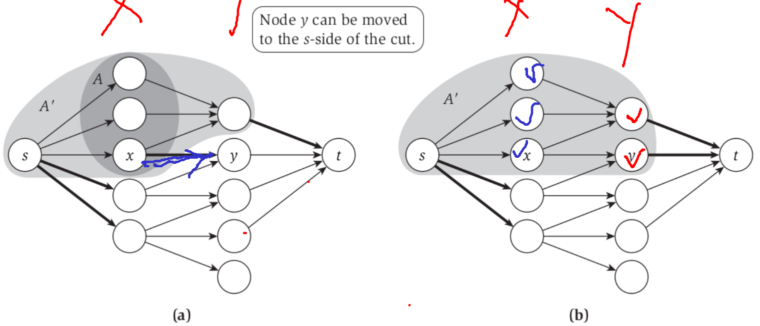
- If $v(f*) = n$ then M_{f*} is a perfect matching. Hence, RHS.
- Conversely If $v(f*) < n$ then we can show that for some $A \subseteq X$ we have $|A| > |E(A)|$.

Proof (cont)



- There exists a min-cut of capacity less than n of type shown above.
- We can see that A is larger than B. (See KT7.5)