# Design and Analysis of Algorithms
# CS218M
## Randomized Algorithms

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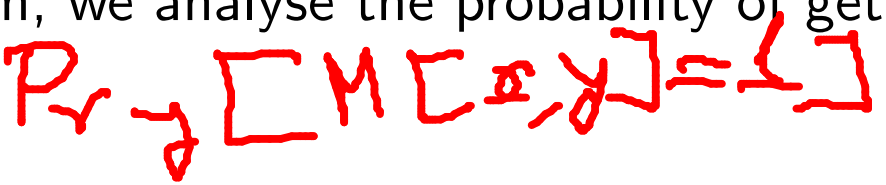

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- As atomic steps, the algorithm uses a random number generator with specified distribution over specified set outcomes.
- Result of the algorithm depends on input as well as the random number generated.
- For a decision problem, we analyse the probability of getting correct answer.
- With repeated trials we can guarantee success with almost 1 probability. We analyse expected running time.

# Approximate MaxSAT

Optimization Problem Given a rectified $3CNF$ formula $\phi$ with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_k$, the aim is to find an assignment which satisfies the maximum number of clauses. Value of solution is number of clauses satisfied. We approximate MaxSAT.

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- Calculate $E(Z) = E(\Sigma_{i=1}^{k} Z_i) = \Sigma_{i=1}^{k}\ E(Z_i) = 7k/8$.

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- What is the expected running time?

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- What is the expected number of independant trials till we get a success?
- Let $R$ be random variable giving number of trials till success. $E[R] = 1 * p + 2 * (1-p) * p + 3 * (1-p)^2 * p + \ldots$
- This simplifies to $E[R] = 1/p$.

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**(13.16)** *There is a randomized algorithm with polynomial expected running time that is guaranteed to produce a truth assignment satisfying at least a $\frac{7}{8}$ fraction of all clauses.*

# Derandomization using conditional expectations

Consider a formula $\phi$ with $k$ clauses where each clause has at most 3 literals (non-overlapping).

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- $E(\phi) = 1/2 \cdot (E(\phi_x) + \#_x)) + 1/2 \cdot (E(\phi_{\neg x}) + \#_{\neg x}))$

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## Algorithm

1. Pick variable $x$.
2. Compute $E_1 = E(\phi)$, $E_2 = (E(\phi_x) + \#_x))$ and $E_3 = (E(\phi_{\neg x}) + \#_{\neg x}))$.
3. If $E_1 < E_2$ set $x = 1$ otherwise $x = 0$.
4. Goto (1) till variables are unassigned.

# Example

- $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$
- $\phi_{x_1} = (1 \vee \neg x_2 \vee \neg x_3) \wedge (0 \vee x_2 \vee x_3) \wedge (1 \vee x_2 \vee x_3)$
  $= (x_2 \vee x_3)$
- $\#_{x_1} = 2.$
- $E(\phi_{x_1}) = (1\ (1/(2^2))) = 3/4.$

$\phi_{\neg x} = (\neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3)$

$\#_{\neg x} = 1$

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- Same as quicksort with modified selection of pivot.
- Central Pivot A pivot which split $S$ in $S^+$ and $S^-$ such that $|S^+| \geq 1/4 \cdot |S|$ and $|S^-| \geq 1/4 \cdot |S|$.
- Repeats random selection of pivot till a central pivot is found. Then partitions the array.

# Algorithm

```
Modified Quicksort(S):
If |S| ≤ 3 then
    Sort S
    Output the sorted list
Endif
Else
    While no central splitter has been found
        Choose a splitter a_i ∈ S uniformly at random
        For each element a_j of S
            Put a_j in S^- if a_j < a_i
            Put a_j in S^+ if a_j > a_i
        Endfor
        If |S^-| ≥ |S|/4 and |S^+| ≥ |S|/4 then
            a_i is a central splitter
        Endif
    Endwhile
    Recursively call Quicksort(S^-) and Quicksort(S^+)
    Output the sorted set S^-, then a_i, then the sorted set S^+
Endif
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# Analysis

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- Hence, recurrence of Expected worst-case execution time of randomized quicksort is roughly $T(n) = T(n/(4/3)) + T(n/(1/4)) + \Theta(n)$.
- Solution using recursion tree is $T(n) = O(n \cdot lg(n))$.

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- Hence for a random choice of pivot $a_i \in S$, the probability that $a_i$ is a central pivot is $p = 1/2$.
- Hence, the waiting time bound states that the expected number of iterations to find the central pivot is $1/p$ i.e. 2.

# Randomized Complexity Classes

## Class $\mathbb{ZPP}$

$L \in \mathbb{ZPP}$ if there exists a randomized algorithm $M(x, y)$ which is polynomial time in expectation such that

- $x \in L \Rightarrow Pr_y[M(x, y) = 1] \geq \alpha$ with $\alpha = 1$.
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- Clearly, $\mathbb{P} \subseteq \mathbb{ZPP} \subseteq \mathbb{RP} \subseteq \mathbb{BPP}$.
- Open Problem: $\mathbb{P} = \mathbb{BPP}$?