Problem 1:(a)[20 marks] Consider the Butterfly network on 2^n rows and $n+1$ columns. Show that its bisection width is $\Omega(2^n)$. Use the technique of embedding a complete directed graph. Clearly state the path used to connect node (i,j) to (k,l).

(b)[10 marks] Consider the forward paths from inputs $x_0, x_1, \ldots, x_{r-1}$ of the Butterfly network above to outputs $0, \ldots, r-1$. Show that the paths are vertex disjoint if $x_0 < x_1 < \cdots < x_{r-1}$.

Problem 2:[20 marks] A medical clinic has the following system for arranging (non-emergency) patients to meet doctors. The entire week is divided into timeslots. Each patient p who needs to see a doctor submits lists D_p, T_p, respectively of preferred doctors and preferred time slots. Each doctor d likewise submits a list A_d of time slots in which he/she is available. For each patient p we must select a time slot t from T_p and a doctor d from D_p such that the selected doctor d is available at t, i.e. $t \in A_d$. Further, each doctor d is assigned an examination room r_d which d must use. Unfortunately, several doctors may be assigned the same examination room; but at a time a room may be used only by one doctor and one patient.

Given D_p, T_p for all p and A_d, r_d for all d, show how to decide whether all patients can be seen. If you cannot solve the problem as stated, but can solve a weaker problem, clearly state that and solve (for appropriate less credit).

Problem 3:(a)[10 marks] Let M, N be matchings in a graph G, with $|M| > |N|$. Show that there must exist matchings M', N' such that $|M'| = |M|-1$, $|N'| = |N|+1$ and the union and intersection of M, N (as edge sets) is the same as that of M', N'.

(b) Let W be the walk matrix of a graph G which consists of two disjoint strongly connected components. (i)[2 marks] What is the largest eigenvalue of W and what is its multiplicity? (ii)[8 marks] Suppose further that each of the 2 strongly connected components are k regular. What is the largest eigenvalue and its multiplicity of the matrix $\begin{pmatrix} W & J/k \\ J/k & W \end{pmatrix}$ where J is the matrix with all entries 1. Explain briefly.

Problem 4: An (α, β, N, d) concentrator has two sets of vertices, U, V, with $|U| = N$ and $|V| = N/2$. The max degree of any node in U is d, and that in V is $2d$. Further every subset of U having $k \leq \alpha N$ nodes is required to be connected to at least βk nodes in V.

(a)[5 marks] Suppose $G_1 = (U_1, V_1, E_1), G_2 = (U_2, V_2, E_2)$ are (α, β, N, d) concentrators. Consider $G' = (U_1 \cup U_2, V_1 \cup V_2, E_1 \cup E_2)$. Is it a concentrator for any choice of parameters?

(b)[5 marks] We studied a randomized algorithm to construct a concentrator. For this algorithm, what is the expected number of nodes in U whose degree will be less than d? You may assume for small x that $(1-x) \approx e^{-x}$.

(c)[5 marks] Suppose graph G results from the randomized algorithm. We proved in class that with finite probability this graph would have β above a suitable threshold. But we did not give any way to check that the actually constructed G had a large β. Suppose the second smallest eigenvalue of the Laplacian of G is computed and found to be μ_2. Show that this can be used to guarantee a lower bound on β. You might find a snippet from Cheeger's theorem useful: " $\frac{r(G)^2 n}{8 d_{max}} \leq \frac{\mu_2}{n} \leq r(G)$ ".

Please Turn Over.

Problem 5:(a)[10 marks] An (m,n) crossbar has m input and n output vertices. In this, it is possible to establish simultaneous connections from input s_i to output t_i for all i where $s_1,\ldots,s_q$ is any sequence of distinct inputs and $t_1,\ldots,t_q$ any sequence of distinct outputs, and $q=\min(m,n)$. An (m,n,r)-Clos network is one recursive implementation of an (nr,nr) crossbar. It is made of (n,m)-crossbars $C_1,\ldots,C_r$, (r,r)-crossbars $D_1,\ldots,D_m$, and (m,n)-crossbars $E_1,\ldots,E_r$. Output j of C_i is input i of D_j. Output i of D_j is input j of E_i. The inputs of the C_is and the outputs of the E_is comprise the inputs/outputs of the Clos network. As was noted in the midsem, $m=n=2$ gives a Benes network.

Consider the case $n=m=3$. Suppose the input i overall is input $i \bmod 3$ of $C_{\lceil i/3\rceil}$. And likewise output i overall is output $i \bmod 3$ of $E_{\lceil i/3\rceil}$. Show that this network can realize any permutation π, i.e. configure the crossbars such that input i is connected to output $\pi(i)$.

Hint: First find r connections to make through D_1.

(b)[5 marks] Let $F=G\square H$. Consider the standard visualization, in which we have G appearing in each column and H in each row. Suppose the vertices of F are numbered in row major order, i.e. vertex ij in the array is numbered $(i-1)n+j$ with m,n the number of vertices in G,H. Let $A(G)$ denote the adjacency matrix of any graph G.

Consider the adjacency matrix $A(F)$. Suppose it is divided into $n\times n$ submatrices. What is the submatrix $A(F)_{ij}$?

Extra credit:[10 marks] Let u,v be eigenvectors of $A(G),A(H)$ of eigenvalues α,β respectively. Let $X=uv^T$ which is an $m\times n$ matrix. Let x be obtained by taking the entries of X in row major order. Show that x is an eigenvector of $A(F)$. What is the eigenvalue for this eigenvector?

Hint: Let $y=A(F)x$, and let x_i,y_i denote the ith set of n values in x,y respectively. Express y_i in terms of $x_1,\ldots,x_j$.