CS 408 **Bisection Lower Bound** **Abhiram Ranade**

Finding the bisection width of a graph is NP-complete. But for nice graphs such as hypercubes, multidimensional arrays and several others, we can compute it exactly. The upper bounds are usually given by inspection. For some graphs such as paths, or the complete binary tree, or the complete graph, lower bounds are easily argued. But in general lower bounds are trickier. It may be possible to make an *exchange argument*, i.e. consider a candidate minimum bisection, argue that getting a new bisection by swapping a pair of vertices should only increase the cost, and using this to get a characterization of the optimal bisection. This argument is often messy, even for simple graphs such as say $P_n \square P_n$. It turns out that it is possible to get lower bounds by looking at the eigenvalues of certain matricess associated with the graph, which we will study later.

A very elegant strategy is based on estimating ratios of bisection widths of graphs.

Theorem 1 *Suppose a graph G on n vertices is embedded in a graph H with load 1, and congestion C. Suppose B_G, B_H denote the bisection widths of G, H respectively. Then $B_H \geq B_G/C$.*

Proof: Let E' denote a set of edges bisecting H into H_1, H_2 such that $|V(H_1)| = \lfloor n/2 \rfloor$, and $|V(H_2)| = \lceil n/2 \rceil$, and $|E'| = B_H$. Consider the subgraphs G_1, G_2 of G induced by the vertices embedded into H_1, H_2 respectively. This is a bisection of G, though not necessarily a minimum bisection. There must be at least B_G edges with one endpoint in G_1 and another in G_2. All such edges must be embedded into paths that pass through E' at least once. Thus the total congestion in E' must be at least B_G. But the congestion of any edge is at most C, hence the total congestion is $C|E'| = CB_H$. Hence $CB_H \geq B_G$, and hence the result follows. ■

Suppose we somehow know the bisection width of G. Then by embedding G into H, we get a lower bound on B_H using the above theorem! The most suitable candidate for embedding is the complete directed on n nodes. The reason for choosing the complete directed graph rather than the complete graph will become clear later.

Corollary 1 *Let H be a graph on n nodes, and B its bisection width. Suppose the complete directed graph on n nodes is embedded into H with load 1, such that the (unidirectional) congestion in each edge is at most C. Then $B \geq \frac{1}{C} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$.*

Proof: The proof of Theorem 1 applies with G the complete directed graph, B_G the size of the bisection counting each edge only in one direction. Let $G_1, G_2, H, H_1, H_2, E', B_H$ be as before. Then we have $B_H \geq B_G/C$. Noting $B_G = \lfloor n/2 \rfloor \lceil n/2 \rceil$ the bound follows. ■

1 Hypercube Bisection Width

We begin by embedding a n node complete directed graph into Q_k, where $n = 2^k$. For this we use canonical paths obtained by correcting bits lsb to msb.

Let u, v be two nodes in Q_k. Let $u = u_{k-1} \ldots u_0$ denote the bits of u, and $v = v_{k-1} \ldots v_0$. Consider the sequence in which the bits of u change to the bits of v, one bit at a time from lsb to msb, i.e. $u_{k-1} \ldots u_0$, $u_{k-1} \ldots u_1 v_0$, $\ldots$, $u_{k-1} v_{k-2} \ldots v_0$, $v_{k-1} \ldots v_0$. Some vertices will appear more than once in this sequence – whenever the corresponding bits of u, v in identical; if these duplicates are removed, then we will have our canonical path from u to v in Q_k. This is how the directed edge in K_n from u to v is embedded. Note that the edge from v to u will be embedded using a different path in general.

So let us estimate the congestion of an edge from $w = w_{k-1} \ldots w_0$ to its neighbour z across dimension i. We will count the number of paths that will use this edge frow w to z.

Suppose a path from u to v uses this edge. Then we know that w must occur in the above sequence given for the vertices on the path from u to v. The only possible place where that path crosses dimension i can be $u_{k-1} \ldots u_i v_{i-1} \ldots v_0$ to $u_{k-1} \ldots u_{i+1} v_i \ldots v_0$. Thus we know that $w = u_{k-1} \ldots u_i v_{i-1} \ldots v_0$. Or alternatively, $w_{k-1} \ldots w_i = u_{k-1} \ldots u_i$, and $w_{i-1} \ldots w_0 = v_{i-1} \ldots v_0$. Note further that $w_i = u_i = \overline{v_i}$. Thus the fact that the path from u to v goes through w constrains how u, v can be chosen. Since the most significant $k - i$ bits of u are required to agree with those of w, there are only 2^i ways to choose the remaining bits for u, and those are the different possible choices for u. The least significant i bits of w, v must agree and the ith least bit must differ, thus there are $k - i - 1$ bits which can be chosen arbitrarily to decide v. Any choice of u, v as described above will cause the path to go over our edge. Thus there are $2^i \cdot 2^{k-i-1} = 2^{k-1}$ choices. But this argument applies to any edge, and hence the congestion is uniformly 2^{k-1}.

Thus the bisection width is $n^2/4C = 2^{2k}/(4 \cdot 2^{k-1}) = 2^{k-1} = n/2$. This is precisely the number of edges along any dimension, and hence there is a matching upper bound as well.

This argument is adequate to give good lower bounds on bisection widths of many, many networks.

Exercises

1. Show that $H = P_r \Box P_c$ where $r \leq c$ has bisection width r without embedding the complete directed graph. *Hint:* Suppose H_1, H_2 is an optimal bisection. Start by arguing that H_1, H_2 need not contain non-consecutive vertices in any column or row.

2. Show that $P_r \Box P_c$ where $r \leq c$ has bisection width r by embedding the complete directed graph. Compare this proof with the preceding one.

3. Show that the converse of Theorem 1 is not true.

4. *(I dont know the answer to this.)* Are there classes of graphs for which the converse might be true? Say perhaps vertex transitive graphs? Say not the exact converse, but something like the converse?

5. Consider the graph obtained by attaching $P_{n^2/3}$ to the center of the longer side of $P_n \Box P_{2n/3}$. Give an upper bound on the bisection width of this graph. Get a lower bound by embedding a complete graph as above. Get a lower bound by embedding a graph G consisting of a $K_{5n^2/6}$ to which is attached $P_{n^2/6}$. Argue a bound on the bisection width of G from first principles.

6. Suppose I want remove minimum number of edges to partition Q_n into one subgraph having 2^k vertices, $k < n$, and the rest. Show that it is possible to do this by removing $2^k(n-k)$ edges and that at least 2^k edges must be removed. The lower bound is also based on embedding the complete directed graph; perhaps it can be improved by embedding some other graph.