Problem 1: Suppose the 2^{2n} node hypercube Q_{2n} is embedded in $P_{2^n} \Box P_{2^n}$ with load 1. Suppose that the nodes of P_{2^n} are numbered 0 to $2^n - 1$.

(a)[5 marks] Give a lower bound on the dilation of the embedding. Hint: consider the dilation of the edges along the path in the hypercube between nodes placed at (0,0) and $(2^n - 1, 2^n - 1)$.

(b)[5 marks] Give a lower bound on the congestion of the embedding.

(c)[5 marks] What is the general principle for the result in part (a)? Only state the principle, you need not prove it.

Problem 2: (a)[10 marks] Show that there are n vertex disjoint paths in Q_n between vertices 0 and $2^n - 1$. Make sure that your answer includes a clear description of the paths (if your proof is constructive) and a proof that the paths are vertex disjoint. (b)[15 marks] Show that there are n vertex disjoint paths between any two nodes u, v of Q_n. For this you may use the result of part (a) even if you did not solve it.

Problem 3: An (m, n) crossbar has m input and n output vertices. In this, it is possible to establish simultaneous connections from input s_i to output t_i for all i where $s_1, \ldots, s_q$ is any sequence of distinct inputs and $t_1, \ldots, t_q$ any sequence of distinct outputs, and $q = \min(m, n)$. An (m, n, r)-Clos network is one recursive implementation of an (nr, nr) crossbar. It is made of (n, m)-crossbars $C_1, \ldots, C_r$, (r, r)-crossbars $D_1, \ldots, D_m$, and (m, n)-crossbars $E_1, \ldots, E_r$. Output j of C_i is input i of D_j. Output i of D_j is input j of E_i. The inputs of the C_is and the outputs of the E_is comprise the inputs/outputs of the Clos network. In the questions below, you will provide a partial proof that the Clos network can indeed establish simultaneous connections as described above.

(a)[5 marks] Show that the Benes network is a special case of the Clos network. What are the values of m, n, r and what play the role of C, D, E?

(b)[10 marks] Suppose $m \geq 2n - 1$. Suppose some $N' < N$ of the $N = nr$ inputs are currently connected to some N' of the nr outputs of the network. Show that if a new request appears on an unused input i of C_j to be connected to some unused output i' of $E_{j'}$, then this can be accomplished *without upsetting any of the present connections*. For this part you may assume that each individual crossbar is capable of adding extra connections between unused inputs-outputs without upsetting the ones present already.

Problem 4: (a)[15 marks] Prove that the $N = 2^n$ node deBruijn graph has bisection width $\Omega(N/\log N)$.

Problem 5:[20 marks] A (α, β, N, d) concentrator has two sets of vertices, U, V, with $|U| = N$ and $|V| = N/2$. For this exercise suppose that the degree of every node in U is exactly d, and that in V is exactly $2d$. Further every subset of U having $k \leq \alpha N$ nodes is connected to at least βk nodes in V. Suppose we are also given that α, β are constants such that $\beta > 1, \alpha\beta > 1/4$.

Show that the diameter of this graph is $O(\log N)$.

Problem 6:(a)[3 marks] What is the maximum number of monochromatic triangles (i.e. subgraphs K_3), not necessarily disjoint, in a 2 colouring of K_{10}?

(b)[7 marks] Show that there exists a 2 colouring of K_{10} with at most 30 monochromatic triangles.