

Let A be an $n \times n$ matrix, and x an $n \times 1$ vector. Then it is useful to bear in mind the following view of matrix vector multiplication $y = xA$.

Think of an n vertex graph, and each x_i resides on vertex i . The matrix entry a_{ij} resides on the directed edge (i, j) . Now the process of multiplication is:

1. All the vertices i send out their entries x_i on all the outgoing edges (i, j) .
2. As an entry travels down an edge, it is multiplied by the entry stored there. Thus $x_i \cdot a_{ij}$ is calculated on edge (i, j) . The calculated value moves towards j .
3. All values received in vertex j are added up to give y_j . Thus we get $y_j = \sum_i x_i a_{ij}$.

This view is especially useful when the graph is sparse, i.e. $a_{ij} = 0$ for many edges.

The multiplication Ax is likewise viewed with values travelling backwards along the edges.

1 Multiplication of stochastic matrices

Suppose A is a right stochastic matrix, i.e. its rows add up to 1 and entries are all non-negative.

Then each entry of $y = Ax$ is a weighted average of the x vector. There are some obvious but important consequences of this: any weighted average can be no larger than the largest element, nor smaller than the smallest element. This immediately proves that the right eigenvalues must be between -1 and 1.¹

Something interesting can also be said for the product $z = xA$. When x_i travels along the edges (i, j) it is multiplied by a_{ij} . Since $a_{i1}, a_{i2}, \dots, a_{in}$ add up to 1, after multiplication the terms $x_i a_{i1}, x_i a_{i2}, \dots, x_i a_{in}$ add up to x_i . So

¹Make sure you understand why the eigenvalues cannot be smaller than -1.

it may be considered that x_i is some kind of mass/charge which is conserved. In the final step, these products are added up. No matter which products get added together, the addition also “conserves mass/charge”. Another way to observe this is to multiply by $\mathbf{1}$ to give $z \cdot \mathbf{1} = xA \cdot \mathbf{1}$. But $\mathbf{1}$ is a right eigenvector of a right stochastic matrix A and hence $z \cdot \mathbf{1} = xA \cdot \mathbf{1} = x \cdot \mathbf{1}$. But multiplication by $\mathbf{1}$ is simply taking the sum; hence $z \cdot \mathbf{1} = x \cdot \mathbf{1}$ essentially expresses the conservation property.