100 marks	**CS 601 Midsemester Test**	**16:30-18:30, 14/9/9**

Problem 1: A graph is said to be d-regular if every vertex has degree d. Consider a d regular bipartite graph $G = (U, V, E)$ with $|U| = |V| = n$.

(a)[7 marks] Let V' denote the set of neighbours of any $U' \subseteq U$. Show that $|V'| \geq |U'|$.

(b)[3 marks] Using above, prove that G has a perfect matching, i.e. of size n.

(a) $|U'|d =$ The number of edges leaving $U' \leq$ number of edges entering $V' = |V'|d$. Thus $|U'| \leq |V'|$.

(b) Let A denote the cut. We can assume there are no edges leaving the cut into V. If there are we could include the vertices they are incident on, as in quiz 1. But then the number of edges in the cut is the number leaving A and the number entering $U - A$.

Or just state that we use Hall's Theorem.

Problem 2:[15 marks] Suppose you are constructing the timetable for the midsemester examinations. Say all exams are of 1 hour duration, and are held in Convocation hall, which has infinite seating capacity, and can seat students of different courses at the same time. The convocation hall has been made available for certain hours on certain days, given as part of the input. You are given the set R_c of students registered for each course c. Clearly, the examinations of the courses taken by any single student must be held at disjoint times. For each course c you are also given a set E_c of experts, one of whom must be present when the examination is held. The same person can simultaneously serve as expert for several examinations, provided he/she is named in their list. Each expert must be paid an honorarium of Rs. 1000 per hour, and there is a total budget Rs. B allocated for this purpose. You are to decide whether the examinations can be held as per the conditions above. Show that this is NP-hard.

Many ways.

Reduction from graph colouring.

F(G,k){ Make a course for each vertex. Make a student for each edge, takes courses corresponding to the endpoints. Convo available for k exams (k=number of colours specified) or Single expert for all courses. Budget = 1000*k. }

Reduction from vertex cover.

F(G,k){ Create one course for each edge. Each vertex is an expert for the incident edges. Budget B = 1000 * target size of the cover.

Give each course one distinct student. Make convo available for just one hour. }

12 marks: getting the general idea, and writing out the proofs both ways and mentioning polytime. 3 marks: ensuring that the irrelevant parameters (e.g. convo availability in second reduction) are assigned in a non interfering manner.

Problem 3: Suppose we are given as input the lengths, widths, and depths $l_i, w_i, d_i, i = 1 \dots n$ for n boxes. We will assume that the box walls have negligible thickness and the boxes are rigid. One or several boxes can be packed inside another box if they fit without crushing. They may be rotated as necessary. Such a box containing other boxes could itself be packed inside another box, and so on. Suppose we are also given as input an integer k.

(a) Consider the problem of deciding whether the boxes can be packed into k boxes (each of which may contain boxes inside boxes and so on). (i)[5 marks] Show that this problem is in NP. Carefully state the certificate that you will use. (ii)[20 marks] Reduce subset sum to this problem. Show how your reduction will map the subset sum instance in which the set of numbers is $\{1, 3, 5, 7\}$ and the target sum is 11. Your answer will not be graded unless you show this mapping.

(i) Certificate must mention where exactly each box is placed inside another. So for each box i you need to give j (the number of the box in which it is to be placed), coordinates of box center, and orientation of x,y,z axes relative to box j. Note that if you just give a list of boxes which go into another box j, it is not easy to verify (in fact determining whether it can be done is NP-hard!). Just giving which boxes go inside which other boxes is not easy to check in polytime. -3 for missing this.

(ii)

$F(x = s_1..s_n, T$: numbers and target$)\{$

Assume s_i are in non-decreasing order. Create n boxes with $l_i = s_i$. $w_i =$ some large number, for all i. The depths are even larger, and almost identical, they reduce slightly as the lengths increase. This ensures that no box can be packed inside another. There are two additional boxes, one of length $= T$, another of length $= (\sum_i s_i) - T$. The width of both is identical to that of the other boxes, and the depth equals the largest depth.

$k = 2$.

Return y= packing instance constructed above.

$\}$ 15 marks.

ss has solution => pack the subset corresponding to s_i in the n+1th box and the rest in the n+2th box. => box packing has solution. 1 mark.

box packing has solution => only last two boxes can be outside of all => what is in n+1th box must have lengths adding up to T. => SS has solution. 3 marks.

1 mark for observing that F runs in polytime.

You get the last 5 marks only if the basic construction is correct.

SS instance: x={1,3,5,7},11. F(x) = k=2, boxes: 1x100x1000, 3x99x1000, 5x98x1000, 7x97x1000, 11x100x1000, 5x100x1000.

Several people did not ensure that the SS instance boxes did not fit inside one another, but got everything else right, including, importantly, stating that k=2. Such answers only got 10 marks (and less if, say, k=2 was not properly stated).

(b)[20 marks] Assume now that the lengths, widths, and depths of all boxes satisfy $0.5 < l_i \leq w_i \leq d_i \leq 1$. Further assume that one box can be placed inside another box only in an axis parallel manner. Notice that this ensures that any box can directly contain at most one other box (which might directly contain at most one, and so on). Show that under this restriction, the problem can be solved in polynomial time.

Very much like airline scheduling. For each box we have an edge. This has capacity 1 and lowerbound

1. From this we have an edge to other boxes that can be packed inside this one. There is a source node that connects to all boxes, and a sink node with demand k.

Problem 4:[30 marks] The meeting scheduling (MS) problem is as follows. A meeting is to be scheduled for each of n committees. Only one meeting can be scheduled each day. Each committee has several members, and a single member may belong to several committees. Say a member is happy if all his meetings are on consecutive days (i.e. there is no gap between meetings). In the optimization problem, the goal is to construct a schedule so as to maximise the number of happy members. The decision problem takes an additional input t and determines whether there exists a schedule in which t members will be happy. Show that MS is NP complete.

NP: give the schedule. Check that every meeting is scheduled, and whether only t members are unhappy. 5 marks.

NPH: reduction from HP. HP ¡=p MS F(G){Vertex becomes a meeting. Edge (u,v) becomes a member who must attend meetings u and v. 10 marks

We want t=n-1 members to be happy. 5 marks }

HP has a solution -¿ schedule meetings in that order, so all the members corresponding to the edges along the path are happy. 3 marks.

MS has a solution with so n-1 happy members. Every pair of meetings can have at most 1 member in common. Hence if there are n-1 in common, they must correspond to one per adjacent pairs of meetings. Thus the same sequence will produce a Hamiltonian Path. 5 marks.

Construction happens in polytime. 2 marks.