60 marks **CS 601 Quiz 2** **6:00-8:00, 23/10/10**

You may reduce from any of the problems we have studied, or any of the problems discussed here, such as the two dominating set problems.

In the *Dominating Set* problem the input is an undirected graph $G = (V, E)$. You are required to find a subset $V' \subseteq V$ of minimum cardinality such that every vertex in $V - V'$ has an edge to some vertex in V'.

In the *Bipartite Dominating Set* problem, the input is a bipartite graph $G = (X, Y, F)$, with X, Y the bipartition of vertices, and F the set of edges. You are required to find a subset $X' \subseteq X$ of minimum cardinality such that every vertex in Y has an edge to some vertex in X.

(1)[10 marks] Show that the bipartite dominating set problem is NP-complete.

NP: The certificate is X'. Just need to check that it is a subset of X, and it has the given cardinality, and that every $y \in Y$ has an edge to some $x \in X'$. This clearly can happen in polytime. 3 marks

Reduce from set cover.

F(Collection of sets C,k){

Consider the bipartite graph representation of set cover in which U' denotes sets, U'' the elements, and edges denote membership. Let $X = U'$. Let $Y = U$. The size of the dominating set needed is k.

Return this instance.

} 4 marks

SC has solution $C' \Rightarrow$ There are k sets in C that cover the universe, i.e. k vertices in U' that have edges to all vertices U'', and also to all vertices in U'. Thus these form the dominating set as required. 1 mark.

If we have a dominating set of size k, then corresponding sets from C form a cover. Hence SC has a solution. 1 mark.

F runs in polytime. 1 mark

(2)[10 marks] Show that bipartite dominating set reduces to dominating set.

F(X,Y,F,k){

The vertex set for the graph G is $V = X \cup Y$. The edge set E includes F, and in addition has edges connecting all vertices in X to each other. The target m for the dominating set size is k.

} 4 marks

BDS has solution X'. X' has an edge to all of Y, but because of the new edges to all of X' as well. Hence X' is a dominating set for G as well, of size $k = m$. 1 mark.

DS has a solution V'. If V' contains any $y \in Y$, then let (x, y) be any edge in F. Redefine $V' = V' - \{y\} \cup \{x\}$. The vertex y only connects to vertices in X, but so does x. Further, x connects to y as well. Hence the new V' is also a dominating set. In this way we remove all vertices in Y from V' and we get a dominating set of the same size as before, contained entirely in X. But this must be a dominating set for the BDS problem X, Y, F, k. 4 marks.

The reduction clearly runs in polytime. 1 mark.

(3)[10 marks] Give a $O(\log |V|)$ approximation algorithm for dominating set.

Consider this as a set cover problem. For each $v \in V$, define a set S_v consisting of v and its neighbours. The universe is V. Now we simply need a set cover of minimum size. This can be done using the greedy algorithm, which is guaranteed to give a cover at most $\log |V|$ times the optimal.

(4)[10 marks] Express the (optimization version of) dominating set problem as an integer program (no proof needed). Suppose now that the graph G is d-regular, i.e. every vertex has exactly d neighbours. Use linear programming rounding to give an $O(d)$ approximate algorithm for finding the (minimum) dominating set.

There will be an indicator variable x_v for every vertex v. 1 will denote that v is in the DS, 0, not in DS. If $v_1, \ldots, v_m$ denote the neighbours of v, we will have an inequality $x_v + \sum_j x_{v_j} \geq 1$, for all vertices v. The objective function will be $\min \sum_v x_v$. 4 marks.

We relax the IP to an LP, i.e. we allow values in the range [0,1]. We set a variable to 1 if it is at least $1/d + 1$. 4 marks.

If G is d regular, then every inequality will have $d + 1$ variables being added. So we know at least one of the variables in each inequality will have to be at least $1/d + 1$. Thus every inequality is satisfied. 1 mark

But every variable is scaled up by a factor $d + 1$ at most. Hence the objective function is also scaled up by a factor $d + 1 = O(d)$. Thus this is an $O(d)$ approximation. 1 mark.

(5)[10 marks] Show that the (metric) p-center problem is NP-complete. The input to the p-center problem is an $n \times n$ matrix D giving distances between n points $C = \{c_1, \ldots, c_n\}$. The distances form a metric, i.e. $d_{ii} = 0$, $d_{ij} = d_{ji}$, $d_{ij} \leq d_{ik} + d_{kj}$ for all i, j, k. The output required is a subset $B \subseteq C$, such that $|B| = p$, and $R = \max_{c_i \in C} \min_{c_j \in B} d_{ij}$ is minimized.

The decision version of the p-center problem has an additional parameter r, and we require $r \geq R$.

F(G,k){ $d_ij = 1$ if (i, j) is in E, 2 otherwise. $r = 1$. $p = k$. } 4 marks.

This runs in polytime. 1 mark.

It produces a metric instance, because $d_{ij} \leq 2 \leq d_{ik} + d_{kj}$. 3 marks

If DS has solution, then every point is at a distance 1 from some point in the DS. Thus pC has a solution. 1 mark

If pC has a solution, then there exist $p = k$ points one of which is at distance 1 from every other point. Thus every vertex has an edge to one of the corresponding vertices. Thus the corresponding vertices form a dominating set of size k. 1 mark.

(6)[10 marks] Show that the non-metric p-center problem does not have a r approximation for any $r > 1$.

Reduce from dominating set, decision version to r-approximation.

F(G,k){

set $d_{ij} = 1$ if $(i, j) \in E$. Otherwise $d_{ij} = rn + 1$.

$p = k$

}

G(G,k,V'){

If V' has radius 1, then V' is a dominating set and so return yes.

If V' has radius $rn+1$, then since the approximation is r, the optimal solution must have radius $rn+1/r > 1$. Thus there arent k points in V which have distance 1 to every one, i.e. there is no dominating set of size k in G. So return no. }