2. $x_1 = [1\,1\,1\,1]^T \qquad x_2 = [1\,1\,-1\,-1]^T \qquad$ ~~both~~ are L.I.

Find vector $\bar{a}$ s.t. $\bar{a}^T x_1 = 0 \qquad \bar{a} = [a_1\ a_2\ a_3\ a_4]^T$

$\bar{a}^T x_2 = 0$

There will be $(4-2=)2$ such L.I. $\bar{a}$.

Many solutions sets exist.

eg: $\bar{a}_1 = [1\ -1\ 0\ 0]^T$

$\bar{a}_2 = [0\ 0\ -1\ 1]^T$

3. Given system of eqns. in $Ax=b$ form will be:

$$\underbrace{\begin{bmatrix} 1+1 & 1+2 & \cdots & 1+n \\ \vdots & & & \\ m+1 & \cdots & \cdots & m+n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ \vdots \\ m \end{bmatrix}}_{b}$$

rank $A = 2$.

$\therefore$ dimension of subspace $\{x \mid Ax = 0\} = n-2$

$\therefore$ Affine dimension $= n-1$.

4. No.

Counter eg: $v_1 = [0, 100] \in S$

$v_2 = [100, 0] \in S$

$\frac{v_1+v_2}{2} = [50, 50] \notin S$.

1. let A be matrix representing orthogonal basis. $\forall A \in O^n$ (space of all $n \times n$ ortho normal matrices)

$\therefore (Ax)^T (Ax) = x^T A^T A x$

$= x^T I x = x^T x$

No this won't happen in 'any' basis as term $A^T A$ will not give I anywhere else. (You can compare coefficients of each term in $x^T A^T A x$ and $x^T x$) and show both are equal iff $A^T A = I$ or $x = \bar{0}$)

5. Boyd.

2.8

(d) $x^T y \leq 1$, $\sum |y_i| = 1$, ($x \succeq 0$ alg also given)

$\Rightarrow$ $x_i \leq 1$ $\forall i$ [Take $y = e_1, e_2 \ldots, e_n$ one-by-one]

$$\therefore \begin{bmatrix} I_{n\times n} \\ -I_{n\times n} \end{bmatrix} x \preceq \begin{bmatrix} 1_{n\times 1} \\ 0_{n\times 1} \end{bmatrix}$$

$\therefore$ Yes

(c) $x^T y \leq 1$

$\Rightarrow$ $\|x\|_2 \|y\|_2 \leq 1$ $\quad$ ($\max x^T y = \|x_2\| \|y\|_2$)

$\Rightarrow$ $\|x_2\|_2 \leq 1$.

$$\therefore \quad x \in R^n_+ \cap \underbrace{\{z \mid \|z\|_2 \leq 1\}_{z \in \mathbb{R}^n}}_{\text{unit ball}}$$

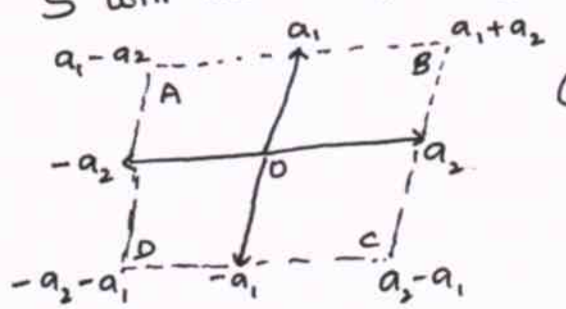
requires intersection of infinitely many half-spaces

$\therefore$ No

(b) Yes. $x \succeq$ $-x \preceq 0$ and 3 given inequalities makes S a polyhedra

(a) Yes

S will be the following parallelogam. ABCD



(assuming a_1 ind. of a_2) else the case is trivial.)

5. Boyd

2.13: XX^T : p.s.d. of rank $= k$

rank$(A+B) \geq$ rank (A,B) if A, B are p.s.d.

$\therefore$ ~~and~~ & also A, B p.s.d $\Rightarrow$ $A+B$ p.s.d

$\therefore \sum\limits_{\substack{X \in S \\ \lambda_i \geq 0}} \lambda_i X X^T$ will be of rank $\geq k$ and $\leq n$

or 0 matrix (if $\lambda_i = 0 \ \forall i$)

p.s.d: positive semi definite.

2.2

Set convex $\Rightarrow$ its intersection with any line convex

Proof.

Set : convex

Line: convex ($\because$ affine)

$\therefore$ Set $\cap$ line will be convex (from Thm)

$\Leftarrow$

Proof:

consider $x, y \in$ Set.(S)

consider Line L passing thru x, y.

given:

Set $\cap$ Line is convex.

also $x, y \in L, S$.

$\therefore$ $\lambda x + (1-\lambda) y \in L, S$. (as intersection is convex).

$\therefore \forall x, y \in S$, $\lambda x + (1-\lambda) y \in S$ $\therefore$ S is convex

Similarily do for affine part.

A6 Suppose convex hulls of X & Y do intersect.

i.e. $\exists\, x \in \mathrm{conv}(X) \cap \mathrm{conv}(Y)$

$$\Rightarrow x = \sum_{i=1}^{k} \lambda_i x_i = \sum_{j=1}^{m} \mu_j y_j \quad \text{— (I)} \qquad \begin{array}{l} \lambda_i \geq 0 \ \sum_i \lambda_i = 1 \\ \mu_j \geq 0 \ \sum_j \mu_j = 1 \end{array}$$

Now at least one of λ_i must be non-zero.

W.l.o.g. let $\lambda_1 \neq 0$, then from (I) we have:

$$x_1 = -\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_1} x_i + \sum_{j=1}^{m} \frac{\mu_j}{\lambda_1} y_j$$

Let $\displaystyle \rho_i = \frac{-\lambda_{i+1}}{\lambda_1} \quad \forall\, i = 1 \text{ to } k-1$

$\displaystyle \rho_{j+k} = \frac{\mu_j}{\lambda_1} \quad \forall\, j = 1 \text{ to } m$

It is easy to see that $\displaystyle \sum_{i=1}^{k+m} \rho_i = 1$

So x_1 is an affine combination of $m+k \geq n+2$ points.

~~Obviously~~ But we know that there can be at most $n+1$ affinely independent pts. in $\mathbb{R}^n$

Hence there can be at most $n+1$ of δ_i are non zero.

i.e. at most $n+1$ of λ_i $i=2$ to k & μ_j, $j=1$ to m can be non-zero. Hence from (I) we have (again w.l.o.g.):

$$\Rightarrow \quad x = \sum_{i=1}^{r_1} \lambda_i x_i = \sum_{j=1}^{r_2} \mu_j y_j \quad \text{where } r_1 + r_2 \leq n+2$$

If $r_1 + r_2 = n+2$, then $\{x_1, \ldots x_{r_1}\}$ & $\{y_1, \ldots y_{r_2}\}$ are exactly the sets whose convex hulls intersect & this cannot happen by the hypothesis that $\text{conv}(X \cap S) \cap \text{conv}(Y \cap S) = \phi$ for S comprised of $n+2$ pts.

If $r_1 + r_2 < n+2$, then $\{x_1, \ldots, x_{r_1}, \underbrace{\ldots}_{\text{some } p_1 \text{ elements of } X}\}$ & $\{y_1, \ldots, y_{r_2}, \underbrace{\ldots}_{\text{some } p_2 \text{ elements of } Y}\}$

such that $r_1 + r_2 + p_1 + p_2 = n+2$.

are the sets whose convex hulls intersect & again this cannot happen.

Hence Proved.

~~(S is non-empty)~~

A7 TST $ri(cl(S)) = ri(S)$

i.e. TST $ri(cl(S)) \subset ri(S)$ & $\underbrace{ri(cl(S)) \supset ri(S)}_{\text{obvious}}$

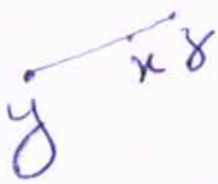
Proof: Let $x \in ri(cl(S))$.

If S is empty then there is nothing to show so assume S is non-empty.

S is non-empty & convex $\Rightarrow ri(S)$ is non-empty.

Pick $y \in ri(S)$

Since $x \in ri(cl(S))$, we can always extend line joining y & x to z, where $z \in cl(S)$.



Now $y \in ri(S)$, $z \in cl(S) \Rightarrow [y, z) \subset ri(S) \Rightarrow x \in ri(S)$

Hence Proved.