Boyd 3.5 :

Non-ve. weighted sum of convex function is convex.
Extending this to ~~infi~~ infinite sums and integrals (Boyd 3.2.1)
if $f(x, y)$ is convex in x for each $y \in A$, $w(y) \geq 0 \ \forall y \in A$

$$g(x) = \int_A w(y) \cdot f(x, y)\, dy$$ is convex (if integral exists)

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt.$$

substitute $t = sx$

$$F(x) = \int_0^1 f(sx)\, ds.$$

as $f(x)$ is convex, $f(sx)$ is convex (simple proof)

$\therefore$ From above, $F(x)$ is convex.

Boyd 3.4 :

Proof by contradiction: Let ~~f~~ 'C' be the upper bound.

$$c \geq f(x) \geq f(x_0) + \underbrace{\nabla f(x_0)^T}_{\text{subdifferential.}} (x - x_0)$$

Let x_0 be the point where $\nabla f(x_0) \neq 0_{n \times 1}$. (if No such point exist. then f will be constant)

take $x = x_0 + \dfrac{(c_1 - f(x_0))\, \nabla f(x_0)}{\|\nabla f(x_0)\|^2}$ where $c_1 > c$.

contradiction happens.

3.12 $A = \text{epi}(f) = \{(x,y) \mid f(x) \leq y\}$ convex

$B = \text{hypo}(g) = \{(x,y) \mid g(x) \geq y\}$ convex (prove - simple)

$ri(A) \cap ri(B) = \emptyset$ as $g(x) \leq f(x)$

$\Rightarrow$ A and B are separable (Sep. Theorem)

$\exists\ a, b$ s.t. $a^T \begin{bmatrix} x \\ y_1 \end{bmatrix} \geq b \quad \forall \begin{bmatrix} x \\ y_1 \end{bmatrix} \in A$

$a^T \begin{bmatrix} x \\ y_2 \end{bmatrix} \leq b \quad \forall \begin{bmatrix} x \\ y_2 \end{bmatrix} \in B$

~~more~~ also $\inf_A a^T \begin{bmatrix} x \\ y_1 \end{bmatrix} \geq b \geq \sup_B a^T \begin{bmatrix} x \\ y_2 \end{bmatrix}$

$a = [a_1 \cdot a_2]$

$\Rightarrow \inf_A a_1^T x + a_2 y_1 \geq b \geq \sup_B a_1^T x + a_2 y_2$

$\Rightarrow \inf_A a_2 y_1 \geq b - a_1^T x \geq \sup_B a_2 y_2$

$\therefore f(x) \geq \underbrace{\frac{b - a_1^T x}{a_2}}_{h(x)} \geq g(x)$ $\quad a_2 \neq 0$ as separating plane is not verticleal.

Proof of why $a_2 \neq 0$

let $a_2 = 0$

then separation thm requires:

$\inf_A a_1^T x + \underbrace{a_2 y_1}_{=0} \geq b \geq \sup_B a_1^T x + \underbrace{a_2 y_2}_{0}$ — ①

and ~~$\sup_A a_1^T x + a_2 y_1 > \inf_B a_1^T x + a_2 y_1 = 0$~~ — ②

condition ~~③~~ ① gets violated as $\inf_{x \in \mathbb{R}^n} a_1^T x < \sup_{x \in \mathbb{R}^n} a_1^T x$.

Boyd 3.31:

a $$g(tx) = \inf_{\alpha>0} t\frac{f(\alpha t x)}{\alpha t} \qquad t \geq 0$$

$$= t \inf_{\beta>0} \frac{f(\beta x)}{\beta}$$

$$= t\, g(x).$$

b $$\frac{h(\alpha x)}{\alpha} \leq \frac{f(\alpha x)}{\alpha} \qquad \alpha > 0$$

$$\Rightarrow \alpha \frac{h(x)}{\alpha} \leq \frac{f(\alpha x)}{\alpha} \qquad \forall \alpha > 0$$

in ~~path~~ particular.

$$h(x) \leq \inf_{\alpha>0} \frac{f(\alpha x)}{\alpha} = g(x)$$

$0 < \lambda < 1$

c $$g(\lambda x) + g((1-\lambda)y)$$

$$= \lambda g(x) + (1-\lambda) g(y)$$

$$\not\geq \inf_{\alpha_1>0} \lambda \frac{f(x\alpha)}{\alpha} + \inf_{\alpha_2>0} (1-\lambda) \frac{f(y\alpha)}{\alpha}$$

$$g(\lambda x) + g((1-\lambda)y) = \inf_{\alpha_1>0} \frac{f(\lambda x \alpha_1)}{\alpha_1} + \inf_{\alpha_2>0} \frac{f((1-\lambda)y\alpha_2)}{\alpha_2}$$

$$\not\geq \inf_{\alpha>0} f(\lambda x$$

$$= \inf_{\beta_1>0} \lambda \frac{f(x\beta_1)}{\beta_1} + \inf_{\beta_2>0} (1-\lambda) \frac{f(y\beta_2)}{\beta_2}$$

as $\min_x (f(x)+g(x)) \geq \min_x f(x) + \min_x g(x)$ $\longrightarrow$ $$\leq \inf_{\beta>0} \lambda \frac{f(x\beta)}{\beta} + (1-\lambda) \frac{f(y\beta)}{\beta}$$

as f is convex $\longrightarrow$ $$\leq \inf_{\beta>0} \frac{f(\beta(\lambda x + (1-\lambda)y))}{\beta}$$

$$= g(\lambda x + (1-\lambda)y)$$

Boyd 3.31c $\quad g(x) = \inf_{\alpha > 0} \dfrac{f(\alpha x)}{\alpha}$

~~$g(\alpha) = \ldots$ is the minimizer, then there is nothing to show. This hints that we~~ Note that, for any convex function f

$$f(x) = \inf_{\mu} \mu \quad \text{s.t. } (x, \mu) \in \operatorname{epi}(f)$$

Also, for any convex set S, if we define

$$g(x) = \inf_{\mu} \mu \quad \text{s.t. } (x, \mu) \in S$$

, then g will be convex

Consider the convex set $\underline{\operatorname{epi}(f) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle/ f(x/\alpha) \le y/\alpha \right\}}$ where $\alpha > 0$

f is convex given

Now g constructed as:

$$g(x) = \inf_{\mu, \alpha > 0} \mu \quad \text{s.t. } (x, \mu) \in \alpha \operatorname{epi}(f)$$

will be convex

$$= \inf_{\mu, \alpha > 0} \mu \quad \text{s.t. } f(x/\alpha) \le \mu/\alpha$$

call $\alpha = 1/\lambda$

$$= \inf_{\mu, \lambda > 0} \mu \quad \text{s.t. } \frac{f(x\lambda)}{\lambda} \le \mu = \underline{\underline{\inf_{\lambda > 0} \frac{f(x\lambda)}{\lambda}}}$$

Boyd.
3.32.

a. $$\frac{d^2 fg}{dx^2} = \underset{\geq 0}{f''}\,\underset{\geq 0}{g} + 2\,\underset{\substack{\geq 0 \\ \text{or} \\ \leq 0}}{f'}\,\underset{\substack{\geq 0 \\ \\ \leq 0}}{g'} + \underset{\geq 0}{f}\,\underset{\geq 0}{g''}$$

$$\geq 0 \Rightarrow \text{convex}$$

b. $$\frac{d^2 fg}{dx^2} = \underset{\leq 0}{f''}\,\underset{\geq 0}{g} + 2\,\underset{\substack{\leq 0 \\ \text{or} \\ \geq 0}}{f'}\,\underset{\substack{\geq 0 \\ \\ \leq 0}}{g'} + \underset{\geq 0}{f}\,\underset{\leq 0}{g''}$$

$$\leq 0 \Rightarrow \text{concave}$$

c $$\frac{d^2 f/g}{dx^2} = \frac{g^2\left[\overset{\geq 0}{f''}\,\overset{\geq 0}{g} - \overset{\leq 0}{g''}\,\overset{\geq 0}{f}\right] - 2\overset{\leq 0}{g'}\left[\overset{\geq 0}{f'}\,\overset{\geq 0}{g} - \overset{\leq 0}{g'}\,\overset{\geq 0}{f}\right]}{g^4}$$

$$\geq 0 \Rightarrow \text{convex}$$

Boyd.
3.35

a. $S_B \subseteq S_{conv(B)}$ trivial of $B \subseteq conv(B)$

$$\begin{aligned}
S_{conv(B)}(y) &= \sup\{y^T x \mid x \in conv(B)\} \\
&= \sup\{y^T \textstyle\sum \lambda_i z_i \mid \sum \lambda_i = 1, \lambda_i \geq 0;\ z_i \in B\} \\
&= \sup\{\cancel{y^T z_i \mid z_i \in B\}}\ \max(y^T z_i) \mid z_i \in B\} \\
&= \sup\{y^T z \mid z \in B\} \\
&= S_B.
\end{aligned}$$

$\underline{b}$ $S_{A+B} = \sup \{ y^T(a+b) \mid a \in A,\ b \in B \}$

$= \sup \{ y^T a \mid a \in A \} + \sup \{ y^T b \mid b \in B \}$

$= S_A + S_B$

$\underline{c}$ $S_{A \cup B} = \sup \{ y^T c \mid c \in A \cup B \}$

$= \sup \{ y^T c \mid c \in A \text{ or } c \in B \}$

$\therefore$ $y^T a^*, y^T b^* \leq y^T c^* \leq \max \{ y^T a^*, y^T b^* \}$

$*$: maximum over the set.

$\therefore$ $S_{A \cup B} = \max \{ S_A, S_B \}$

as $\max\{\ ,\ \} \leq y^T c^* \leq \max\{\ ,\ \}$

$\Rightarrow$ $y^T c^* = \max \{ y^T a^*, y^T b^* \}$

$\underline{d}$ $A \subseteq B \Rightarrow S_A(y) \leq S_B(y)$ trivial

For $S_A(y) \leq S_B(y) \Rightarrow A \subseteq B$ consider ~~contradiction~~. Contrapositive

$\therefore$ T.pt. $A \nsubseteq B \Rightarrow \exists y \text{ s.t. } S_A(y) > S_B(y)$

let $z \in A$, $z \notin B$.

$\therefore$ $\exists$ separating hyperplane. bet^n z and B.

$y^T z \geq d$. $z \in A$, $z \notin B$

$y^T b \leq d \quad \forall b \in B$

$\exists y_1$ s.t. $y_1^T z > y_1^T b \quad \forall b \in B$

$> \sup \{ y_1^T b \mid b \in B \}$

$\therefore$ $\sup \{ y_1^T a \mid a \in A \} \geq y_1^T z > \sup \{ y_1^T b \mid b \in B \}$

$\Rightarrow \exists\ y_1$ s.t. $S_A > S_B$

10. LP $\min_{x,y,z,a,b,c} \quad a+b+c$

s.t.

$$\left.\begin{aligned} a &\geq x \\ a &\geq -x \end{aligned}\right\} |x|$$

$$\left.\begin{aligned} b &\geq y \\ b &\geq -y \end{aligned}\right\} |y|$$

$$\left.\begin{aligned} c &\geq z \\ c &\geq -z \end{aligned}\right\} |z|$$

$$x+y \leq 1$$

$$2x+z=3$$

~~Dual~~: Write it in the for.

$$\max_{w} \quad d^T w \qquad w = \begin{bmatrix} x \\ y \\ z \\ a \\ b \\ c \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$Aw \leq b$$

~~find~~. Calculate A, b.

Dual. $\min_{v} \quad b^T v$

$$A^T v = d$$

$$v \geq 0$$

9 Farkas Lemma: See. Nemiroski book