Quiz-2 (CS709)

Total Marks: 10. Duration: 1hr.

Consider the following convex program[1]:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x, \\ \text{s.t.} \quad & b - A^\top x \in K, \end{aligned}$$

where $c \in \mathbb{R}^n, b \in \mathbb{R}^m$, A is an $n \times m$ matrix and $K \subset \mathbb{R}^m$ is a cone. Assume that the program is feasible. Given this, attempt the following:

1. Prove this statement: the above program is bounded $\Rightarrow \exists y \in \mathbb{R}^m \ni Ay + c = 0$.

   [3 Marks]

2. In general, is the converse of the above statement true?

   [1 Mark]

3. Fill in the blanks[2] in this statement: x^* is optimal for the program if and only if there exists a $y^* \in \mathbb{R}^m$ such that:

   (a) -------------

   (b) $Ay^* + c = 0$

   (c) -------------

   Note that you need to fill each of the above blank with exactly one expression and moreover in such a fashion that once you consider the special case $K = \mathbb{R}^m_+$ (i.e., the program is a linear program[3]), the above conditions must turn out to be exactly the same as the KKT conditions for this case.

   [6 Marks]

[1] We will hence-forth refer to programs of this form as conic programs.

[2] Needless to say, you need to provide rigorous justification for the blanks you fill.

[3] $\mathbb{R}^m_+ \equiv \{x \in \mathbb{R}^m \mid x \geq 0\}$.

Quiz 2 Solutions

19/10/12

Given the following feasible conic program:

$$(P) = \min_{x \in \mathbb{R}^n} c^T x$$
$$\text{s.t. } b - A^T x \in K$$

(1) TST (P) is bounded $\Rightarrow \exists y \in \mathbb{R}^m \ni Ay + c = 0$

i.e. TST (P) is bounded $\Rightarrow c \in \mathcal{C}(A)$ (the column space of A)

If $c = 0$, there is nothing to prove, so let's assume $c \neq 0$.

Proof is by contradiction. Suppose $c \notin \mathcal{C}(A)$. Then we have that $\exists\, v \in \mathcal{N}(A^T) \ni c^T v \neq 0$ and in particular $c^T v < 0$.
(null space of A^T) (by rank-nullity theorem)

Now since (P) is feasible, we have $\exists$ a $x_0 \in \mathbb{R}^n \ni b - A^T x_0 \in K$

$\Rightarrow b - A^T(x_0 + \lambda v) \in K \quad \forall \lambda \geq 0$

Consider the sequence ~~$c^T v, 2c^T v, \ldots$~~
$c^T(x_0 + v),\ c^T(x_0 + 2v),\ c^T(x_0 + 3v),\ \ldots\ldots$

This goes to $-\infty$ as $c^T v < 0$. Since all $x_0 + \lambda v$ are feasible, we have that $(P) = -\infty$ i.e. (P) is unbounded.

Hence Proved.

(2) The converse is not true in general.

For eg. ~~the~~ (P) is unbounded whenever $K = \mathbb{R}^m$, irrespective of the relation between c & A.

(3) x^* is optimal $\Leftrightarrow$ $c^T x \geq c^T x^* \quad \forall \; x \ni b - A^T x \in K$

$\Leftrightarrow$ (P) is bounded & $c^T x \geq c^T x^* \; \forall \; x \ni b - A^T x \in K$

$\Leftrightarrow \exists \; \lambda^* \in \mathbb{R}^m \ni A\lambda^* + c = 0, \; c^T x \geq c^T x^* \; \forall \; x \ni b - A^T x \in K$

$\Leftrightarrow \exists \; \lambda^* \in \mathbb{R}^m \ni A\lambda^* + c = 0, \; -\lambda^{*T} A^T x \geq -\lambda^{*T} A^T x^* \; \forall \; x \ni b - A^T x \in K$

$\Leftrightarrow \exists \; \lambda^* \in \mathbb{R}^m \ni A\lambda^* + c = 0, \; \lambda^{*T}(b - A^T x) \geq \lambda^{*T}(b - A^T x^*) \; \forall \; x \ni b - A^T x \in K$

$\Leftrightarrow \exists \; \lambda^* \in \mathbb{R}^m \ni A\lambda^* + c = 0, \; \min_{\substack{x \in \mathbb{R}^n \\ \text{s.t. } b - A^T x \in K}} \lambda^{*T}(b - A^T x) = \lambda^{*T}(b - A^T x^*)$

Now, $\min_{\substack{x \in \mathbb{R}^n \\ \underline{\text{s.t.}} \; b - A^T x \in K}} \lambda^{*T}(b - A^T x) = \begin{cases} 0 & \lambda^* \in K^* \\ -\infty & \lambda^* \notin K^* \end{cases}$

Hence, x^* is optimal $\Leftrightarrow \exists \; \lambda^* \in \mathbb{R}^m \ni A\lambda^* + c = 0, \; \lambda^* \in K^*, \; \lambda^{*T}(b - A^T x^*) = 0$

~~Northern~~ In case $K = \mathbb{R}^m_+ \longrightarrow$ (same) $\quad (\lambda^* \geq 0) \quad \left(\lambda^*_i (b - A^T x^*)_i = 0 \;\; \forall i \right)$

(because x^* is feasible.

KKT conditions in LP case!

So from now on we will call the ~~above~~ as $\underline{\text{KKT}}$ conditions.