# Mid-Semester Exam (CS-709)

15-Sep-2012

**Note: There are two sections, totally 7 questions. Max. marks=15. Duration=2hrs.**

## 1 Determining Convexity

**Note:** Prove each of the following statements:

1. The negative harmonic mean function, $f : \mathbb{R}^n_{++} \mapsto \mathbb{R}$ given [1] by $f(x) = \frac{-n}{\frac{1}{x_1}+\ldots+\frac{1}{x_n}}\ \forall\ x_i > 0,\ \forall\ i = 1, \ldots, n$, is convex.

   **[1.5 Marks]**

2. Let $C \subset \mathbb{R}^n$ be a non-empty set. The function $f : \mathbb{R}^n \mapsto \mathbb{R}$ given by $f(x) =$ the distance between $x$ and the farthest point in $C$ from $x$, is convex.

   **[1.5 Marka]**

3. The function $f : \mathbb{R}^n \mapsto \mathbb{R}$ given by $f(x) =$ sum of $r$ largest components/entries of $x$. Here $r$ is a given number between 1 and $n$.

   **[2 Marks]**

4. The function $f : S^n_{++} \mapsto \mathbb{R}$ given[2] by $f(M) = trace(M^{-1})\ \forall\ M \succ 0$ is sub-differentiable everywhere.

   **[2 Marks]**

[1] $\mathbb{R}^n_{++} = \left\{x = [x_1\ \ldots\ x_n]^\top \mid x_i > 0\ \forall\ i = 1, \ldots, n\right\}$.
[2] $S^n_{++} = \{X \in S^n \mid X \succ 0\}$

## 2 True or False Questions

**Note:** Respond to each of the following statements by indicating True(Yes) or False(No). Also, in case of each statement, provide a short illustrative or intuitive argument followed by a rigorous proof/answer to support your response.

1. The value of a conic function may be negative at some point in the domain; however the same cannot happen for a function that is dual of a conic function.

   **[2 Marks]**

2. Let $f : C \mapsto \mathbb{R}$ be a closed convex function and $f^{'} : \bar{C} \mapsto \mathbb{R}$ be its conjugate. Being crazy, suppose I share with you $f^{'}$ alone and hide $f$ completely, and then ask you to compute the sub-differential set of $f$ at **any** $x_0 \in C$. Can you do it[3]? i.e., can you write some expression involving $f^{'}$ (and not $f$) that gives you the sub-differential set of $f$ at any $x_0 \in C$?

   **[4 Marks]**

3. Is an analogous statement of the above for the case of conic functions true? i.e., if $f : K \mapsto \mathbb{R}$ is a conic function and $f^{*} : \bar{K} \mapsto \mathbb{R}$ is its dual. Then can you write some expression involving $f^{*}$ (and not $f$) that gives you the sub-differential set at **any** $x_0 \in K$?

   **[2 Marks]**

[3] If any of you answer as 'Yes', then I am not crazy :)

# MidSem Exam Solutions

15/Sep/2012

## Section I

① Easiest way is ~~through~~ by considering 1-d restriction of the given function.

Consider any line $\underset{\in\mathbb{R}^n}{x}+\underset{\in\mathbb{R}}{t}\,\underset{\in\mathbb{R}^n}{y}$ such that $x+ty>0 \;\forall\; t\in J$

Consider $g: J\to\mathbb{R}$ given by $g(t)=f(x+ty)$

$$= \frac{-n}{\frac{1}{x_1+ty_1}+\dots+\frac{1}{x_n+ty_n}}$$

$$\frac{dg(t)}{dt} = n\left(\frac{\sum_i \frac{y_i}{(x_i+ty_i)^2}}{\left(\sum_i \frac{1}{x_i+ty_i}\right)^2}\right)$$

$$\frac{d^2g(t)}{dt^2} = -n\left(\frac{-2\left(\sum_i \frac{y_i^2}{(x_i+ty_i)^3}\right)\left(\sum_i \frac{1}{x_i+ty_i}\right)^2 + 2\left(\sum_i \frac{y_i}{(x_i+ty_i)^2}\right)^2\left(\sum_i \frac{1}{x_i+ty_i}\right)}{\left(\sum_i \frac{1}{x_i+ty_i}\right)^4}\right)$$

$$= \underbrace{\boxed{\frac{2n}{\left(\sum_i \frac{1}{x_i+ty_i}\right)^3}}}_{>0}\left[\underbrace{\left(\sum_i \frac{y_i^2}{(x_i+ty_i)^3}\right)\left(\sum_i \frac{1}{x_i+ty_i}\right) - \left(\sum_i \frac{y_i}{(x_i+ty_i)^2}\right)^2}_{\geq 0}\right]$$

by Cauchy-Schwartz inequality.

Let $u_i = \frac{1}{\sqrt{x_i+ty_i}}$ ; $v_i = \frac{y_i}{(x_i+ty_i)^{3/2}}$

$\|u\|^2\|v\|^2-(u^Tv)^2 \geq 0$

$\Rightarrow g$ is convex ~~(Strictly~~ $\Rightarrow f$ is convex. Hence Proved.

(2) $$f(x) = \max_{y \in C} \|x-y\|$$

Easiest way is to show epi(f) is convex.

$$\text{epi}(f) = \left\{ (x,z) \;\middle|\; \max_{y \in C} \|x-y\| \le z \right\}$$

$$= \left\{ (x,z) \;\middle|\; \underbrace{\|x-y\| \le z}_{} \;\; \forall y \in C \right\}$$

for y fixed this is a convex set, infact a shifted ice-cream cone.

Hence epi(f), which is intersection of ↑ is convex ⇒ f is convex (infact, closed convex)

(3) Easiest way is by writing f as max of affine/linear functions.

Consider functions $g_i(x)$ = sum of <u>any</u> $r$ components of $x$

there will be only ${}^{n}C_{r}$ such distinct functions.

It is easy to see that $f(x) = \max_{i \in I} g_i(x)$

↓ all linear functions

⇒ f is convex ⇒ f is convex.

(4) We need to show $\exists\ \nabla f(M_0)$ (for any $M_0 > 0$)

such that

$$\text{trace}(M^{-1}) \geq \text{trace}(M_0^{-1}) + \text{trace}\left(\nabla f(M_0)(M - M_0)\right)$$

i.e. goal is to ~~find~~ guess a sym. matrix $\nabla f(M_0)$ ~~$\nabla f(M_0)$ where~~ where,

$$\text{trace}\left(M^{-1} - M_0^{-1} - \nabla f(M_0)(M - M_0)\right) \geq 0 \quad \forall\ M, M_0 > 0$$

obvious idea is to choose a matrix $\nabla f(M_0)$ such that we get a square term inside the trace.

such a choice is $\nabla f(M_0) = -M_0^{-2}$, because

$$\text{trace}\left(M^{-1} - M_0^{-1} + M_0^{-2}(M - M_0)\right) = \text{trace}\left(M^{-1/2} - M_0^{-1} M^{1/2}\right)^2 \geq 0 \quad \forall\ M, M_0 > 0$$

$$\therefore\ \nabla f(M_0) = -M_0^{-2}$$

## Section II

(1) This was actually done during lectures. So I won't repeat.

(2) Intuitive answer: Of course yes, because given $f'$, its conjugate, which is $f$, is determined. Here we should be able to compute sub-differential of $f$ via $f'$ alone.

The required expression:

intuition from supporting hyperplane defn.

$$\underset{\downarrow \atop S_1 \text{ (say)}}{\partial f(x_0)} = \underset{x \in \bar{C}}{\operatorname{argmax}} \; \langle x, x_0 \rangle - f'(x)$$

$S_2$ (say)

(i) TST $S_1 \subset S_2$

Let $\nabla f(x_0)$ be a sub-gradient, then we need to show $\nabla f(x_0) \in S_2$.

$$\Updownarrow$$

$$f(x) \geq f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle \quad \forall x \in C$$

$$f(x) - \langle \nabla f(x_0), x \rangle \geq f(x_0) - \langle \nabla f(x_0), x_0 \rangle \quad \forall x \in C$$

$$\Leftrightarrow \quad f(x) - \langle \nabla f(x_0), x \rangle = f(x_0) - \langle \nabla f(x_0), x_0 \rangle$$

$$\Leftrightarrow \quad \max_{x \in C} f(x) - \langle \nabla f(x_0), x \rangle = f(x_0) - \langle \nabla f(x_0), x_0 \rangle$$

($\because$ above inequality is strict at $x = x_0$)

$$\Leftrightarrow \quad f'(\nabla f(x_0)) = f(x_0) - \langle \nabla f(x_0), x_0 \rangle$$

$$\Leftrightarrow \quad f(x_0) = \max_{x \in \bar{C}} \langle x, x_0 \rangle - f'(x) = \langle \nabla f(x_0), x_0 \rangle - f'(\nabla f(x_0))$$

$$\Rightarrow \quad \nabla f(x_0) \in S_2$$

(ii) The proof for $S_2 \subset S_1$ is exactly the same as above with reversed implications.

(3) Since we already know an expression relating $f'$ & sub-gradients of $f$, the idea here is to relate $f^*$ & $f'$ and then $f^*$ & sub-grad. of $f$ are then related.

$$f'(x) = \max_{y \in K} \langle x, y\rangle - f(y)$$

(epigraph trick)

$$= \max_{y \in K, \alpha \in \mathbb{R}} \langle x, y\rangle - \alpha \quad \underline{\underline{\text{s.t.}}} \quad f(y) \le \alpha$$

$(\because f \text{ is conic})$

$$= \max_{y \in K, \alpha \in \mathbb{R}} \alpha(\langle x, y\rangle - 1) \quad \underline{\underline{\text{s.t.}}} \quad f(y) \le 1$$

$$= \begin{cases} \infty & \text{if } \underbrace{\boxed{\max_{y \in K} \langle x, y\rangle \;\; \underline{\text{s.t.}} \;\; f(y) \le 1}}_{= f^*(x)} > 1 \\ 0 & \text{if } f^*(x) \le 1 \end{cases}$$

Now, $\partial f(x_0) = \operatorname{argmax}_{x \in \bar{K}} \langle x, x_0\rangle - f'(x)$

$$= \operatorname{argmax}_{x \in \bar{K}} \langle x, x_0\rangle \quad \text{s.t.} \quad f^*(x) \le 1$$

$\rightarrow$ reqd. expression.