Quiz-1 (CS-709)

17-Aug-2012

Note: *Please provide* rigorous and short *answers. You should use definitions appearing in Notes alone. In case you use some results/theorems from the Notes/Books, then please indicate so. Needless to say, you may start attempting problems by intuitively imagining the set-up of the problem and guessing the results in it. The first question is compulsory*[1]. Duration: 1hr.

1. Did you work out the: i) problem-set for this and/or previous years? ii) Quiz/exam problems from previous years iii) exercise problems from Boyd and/or other books? Be honest as this question carries zero marks.

2. Let set $S \subset V$ be a non-empty set of vectors lying in $\mathcal{V} = (V, +, \cdot, \langle\rangle)$, a finite-dimensional Hilbert space. Let $v \notin S$ be another vector. Consider the set $C = \{x \mid ||x - v|| \leq ||x - s|| \; \forall \; s \in S\}$. Is this set linear/affine/cone/convex/polytope?

[4 Marks]

3. Consider the set $K = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}, ||x_1||_M \leq x_2 \right\}$, where[2] $M \succ 0$ (positive-definite matrix) and $n \in \mathbb{N}$. Is this set linear/affine/cone/convex? Irrespective of your answer, compute the dual cone i.e., give a highly simplified expression for the description of the dual cone. Can you visualize such dual cones (give some illustrations)?

[6 Marks]

[1]Your answer script will NOT be evaluated if you don't answer the first question.

[2]Recall the definition: $\|x\|_M \equiv \sqrt{x^\top M x}$.

4. [**BONUS question to be done after the quiz (5marks)**]. Using any results in books you read and rough-sketch of proof given in lecture, show that: K^* is polyhedral, whenever K is.

Quiz I Solutions (CS709) . 17/Aug/2012.

(2) $C = \{ x \mid \|x-y\| \leq \|x-s\| \quad \forall s \in S \}$

$= \{ x \mid \|x-y\|^2 \leq \|x-s\|^2 \quad \forall s \in S \}$ (∵ Non-negativity of norm)

$= \{ x \mid \|x\|^2 + \|y\|^2 - 2\langle x,y \rangle \leq \|x\|^2 + \|s\|^2 - 2\langle s,x \rangle \quad \forall s \in S \}$ (∵ Again, inner-product defn.)

$= \{ x \mid \langle y-s, x \rangle \geq \underbrace{\frac{\|y\|^2 - \|s\|^2}{2} \quad \forall s \in S \}}$ "

For each $s \in S$ is a half-space.

(2 marks)

Hence C is intersection of half-spaces. It is easy to verify that this is a convex set:

$x_1 \in C \Rightarrow \langle y-s, x_1 \rangle \geq \frac{\|y\|^2 - \|s\|^2}{2} \quad \forall s \in S$ — (1)

$x_2 \in C \Rightarrow \langle y-s, x_2 \rangle \geq \frac{\|y\|^2 - \|s\|^2}{2} \quad \forall s \in S$ — (2)

(1 mark)

Taking $\rho(1) + (1-\rho)(2)$ gives that $\rho x_1 + (1-\rho)x_2 \in C$.
(for $\rho \in [0,1]$)

In general, this set need not even contain the origin/identity and hence is not linear & cone. It is easy to give examples of S where C is a cone (eg. S is singleton $\{-y\}$). ~~However, the set is never linear or affine as~~ Similarly we can give example cases where C is linear/affine: (eg. Take S as any line through y except y)

(1 mark)

$2+1+1=4$

(3) It is easy to show that K is a cone:

$$\begin{rmatrix} x_1 \\ x_2 \end{rmatrix} = x \in K \Rightarrow \|x_1\|_M \le x_2 \quad \Big\} \Rightarrow \quad \rho\|x_1\|_M \le \rho x_2 \quad \forall \rho, \mu \ge 0$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = x' \in K \Rightarrow \|x_1'\|_M \le x_2' \qquad \mu\|x_1'\|_M \le \mu x_2$$

$$\Downarrow$$

$$\|\rho x_1\|_M + \|\mu x_1'\|_M \le \rho x_2 + \mu x_2 \;\; \forall \rho,\mu \ge 0 \Leftarrow \begin{array}{l} \|\rho x_1\|_M \le \rho x_2 \\ \|\mu x_1'\|_M \le \mu x_2 \end{array} \quad \forall \rho, \mu \ge 0 \quad (\because \|\cdot\|_M \text{ is a norm proved in lectures})$$

$$\Downarrow$$

$$\|\rho x_1 + \mu x_1'\|_M \le \rho x_2 + \mu x_2 \quad \forall \rho, \mu \ge 0 \qquad (\because \text{ by triangle inequality satisfied by } \|\cdot\|_M \text{ norm})$$

$$\Downarrow$$

K is closed under conic combinations and hence a cone.

1 mark

This set is never a affine/linear set as: It contains any vector of the form $\begin{bmatrix} 0 \\ \delta \end{bmatrix} \begin{matrix} \in \mathbb{R}^n \\ \in \mathbb{R}^+ \end{matrix}$, however does not have $\begin{bmatrix} 0 \\ -\delta \end{bmatrix}$.

$$K^* = \left\{ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \;\middle|\; y_1 \in \mathbb{R}^n, y_2 \in \mathbb{R}, \;\; y^T x \equiv y_1^T x_1 + y_2 x_2 \ge 0 \;\forall x \in K \right\}$$

Let $M = L \Lambda L^T$ be the eigen-value-decomposition

Lets denote $\Lambda^{1/2} L^T x_1 \equiv z_1$, then, (note that this gives $x_1 = L\Lambda^{-1/2} z_1$)

$$K^* = \left\{ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^{\in \mathbb{R}^n}_{\in \mathbb{R}} \;\middle|\; y_1^T L \Lambda^{-1/2} z_1 + y_2 x_2 \ge 0 \;\; \forall z_1, x_2 \text{ such that } \|z_1\| \le x_2 \right\}$$

By Cauchy-Schwartz inequality we know

$$y_1^T L \Lambda^{-1/2} z_1 \ge -\|\Lambda^{-1/2} L^T y_1\| \, \|z_1\| \text{ \& more}$$

importantly the inequality is achieved (when z_1 & $\Lambda^{-1/2} L^T y_1$ are anti-parallel)

Hence, since we are looking for y such that

$$y_1^T L \Lambda^{-1/2} z_1 \geq -y_2 x_2 \quad \forall \; z_1, x_2 \text{ such that } \|z_1\| \leq x_2,$$

we obtain from Cauchy-Schwartz that:

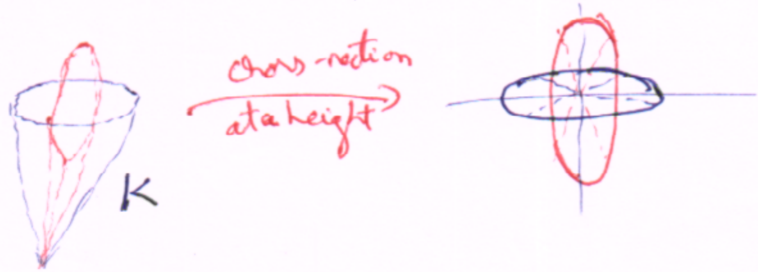
$$K^* = \left\{ y \;\middle|\; -\|\Lambda^{-1/2} L^T y_1\| \, x_2 + y_2 x_2 \geq 0 \quad \forall \; x_2 \geq 0 \right\}$$

$$= \left\{ y \;\middle|\; \left(y_2 - \sqrt{y_1^T M^{-1} y_1}\right) x_2 \geq 0 \quad \forall \; x_2 \geq 0 \right\}$$

3 marks

$$= \left\{ y \;\middle|\; \|y_1\|_{M^{-1}} \leq y_2 \right\}.$$

ILLUSTRATIONS:



2 marks for construction of figures

$1 + 3 + 2 = 6$ marks