Quiz-1 (CS-709)

19-Aug-2011

Note: Please provide *rigorous and short* answers.

1. Consider two inner-products spaces: $\mathcal{I}_1 = (V_1, +_1, \cdot_1, \langle\rangle_1)$ and $\mathcal{I}_2 = (V_2, +_2, \cdot_2, \langle\rangle_2)$. Consider the inner-product space $\mathcal{I}$, which is the direct sum of them i.e., $\mathcal{I} = \mathcal{I}_1 \oplus \mathcal{I}_2$. Consider a closed cone $C_1 \subset V_1$ and another closed cone $C_2 \subset V_2$. Is the Cartesian product[1] of these two cones i.e., $C_1 \times C_2$ a cone in $\mathcal{I}$? Can the dual cone of this Cartesian product, $(C_1 \times C_2)^*$, be written in terms of dual cones C_1^* and C_2^* ? If so, how ? Please justify.

   [4 Marks]

2. Consider the set of all **symmetric** matrices with a given dimension $n \times n$ (denoted by S^n) endowed with the usual $+, \cdot, \langle\rangle_F$ to form an inner-product space. The following conic-section in this space was taken as an example in the lectures:

   $$\mathcal{C} = \{M \mid M \text{ is psd and } trace(M) = 1\}\,.$$

   What is the dimension of this conic-section? Give a simple description, which is different from the dual description, of the polar-set of this conic-section. What is the dimension of this polar set?

   [6 Marks]

[1] Recall the school days definition of Cartesian product of two sets X, Y as $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

19/Aug/2011

Quiz I Solutions

Ans1 ~~TST~~ The simplest way to imagine is consider I_1 as the x-y plane & I_2 as x-z plane; C_1 as a simple 'V' shape cone in x-y plane & C_2 as a simple 'V' shape cone in x-z plane.

With this example we conjencture the following:

(i) $C_1 \times C_2$ is a cone (ii) $(C_1 \times C_2)^* = C_1^* \times C_2^*$

Proof (i)

$v \in C_1 \times C_2 \Rightarrow v = (v_1, v_2)$ where $v_1 \in C_1$, $v_2 \in C_2$

$w \in C_1 \times C_2 \Rightarrow w = (w_1, w_2)$ where $w_1 \in C_1$, $w_2 \in C_2$

$\Rightarrow \lambda \cdot_1 v_1 +_1 \varsigma \cdot_1 w_1 \in C_1$

$\Rightarrow \lambda \cdot_2 v_2 +_2 \varsigma \cdot_2 w_2 \in C_2$

(for $\lambda, \varsigma \geq 0$)
as C_1 & C_2 are cones

$\Rightarrow \lambda \cdot v + \varsigma \cdot w \in C_1 \times C_2$

[where $+$, $\cdot$ are operators for I. This comes from defn. of direct sum]

As $C_1 \times C_2$ is closed under conic comb., it must be a cone.

Proof (ii) $x \in (C_1 \times C_2)^* \iff \langle x, c \rangle \geq 0 \ \forall c \in C_1 \times C_2$ [here $\langle\,\rangle$ is operator for I]

by defn. of dual cone

$\iff \langle x_1, c_1 \rangle_1 + \langle x_2, c_2 \rangle_2 \geq 0 \ \forall c_1 \in C_1 \ \& \ c_2 \in C_2$

by defn. of direct sum

[here $x = (x_1, x_2)$, $c = (c_1, c_2)$]

This is our claim (I)

$\iff \langle x_1, c_1 \rangle_1 \geq 0 \ \forall c_1 \in C_1$ & $\langle x_2, c_2 \rangle_2 \geq 0 \ \forall c_2 \in C_2$

$\iff x_1 \in C_1^*, \ x_2 \in C_2^*$ (Hence proof of (ii) is complete)

Now lets prove claim (I): $\Leftarrow$ is obvious.

~~Proof~~ $\Rightarrow$: TST $\langle x_1, c_1 \rangle_1 + \langle x_2, c_2 \rangle_2 \geq 0 \Rightarrow$ $\langle x_1, c_1 \rangle_1 \geq 0 \ \forall c_1 \in C_1$, $\langle x_2, c_2 \rangle_2 \geq 0 \ \forall c_2 \in C_2$

This is also easy ~~Be~~ $\rightarrow$ Since C_1 is a cone $O \in C_1$

$$\therefore LHS \Rightarrow \langle x_1, 0\rangle_1 + \langle x_2, c_2\rangle_2 \geq 0 \quad \forall c_2 \in C_2$$
$$\Rightarrow \langle x_2, c_2\rangle_2 \geq 0 \quad \forall c_2 \in C_2$$

IIIly since C_2 is a cone, $O \in C_2$

$$\therefore LHS \Rightarrow \langle x_1, c_1\rangle_1 + \langle x_2, 0\rangle_2 \geq 0 \quad \forall c_1 \in C_1$$
$$\Rightarrow \langle x_1, c_1\rangle_1 \geq 0 \quad \forall c_1 \in C_1$$

Marking Scheme for Q1: 1 mark for guessing conjecture (ii)
1 mark for guessing & proving conjec. (i)
2 marks for formally proving (ii)

Ans 2 ~~Since~~ We know that closure of all pds = all psds $\therefore$ The set of all psds has non-empty interior. Hence dimension of $\mathcal{C}$ must be dimensionality of the affine set trace$(M)=1$, which infact is a hyperplane. Therefore dimensionality of $\mathcal{C}$ is one less than that of S^n

$$\Rightarrow \dim(\mathcal{C}) = \frac{n(n+1)}{2} - 1 = \frac{n^2+n-2}{2}$$

$$\mathcal{C}^* = \left\{ M^{\in S^n} \middle| \langle M, C\rangle_F \leq 1 \ \forall C \in \mathcal{C} \right\}$$

Now each $C \in \mathcal{C}$ can be written as $\sum_i \lambda_i x_i x_i^T$ where $\overbrace{\lambda_i \geq 0 \ \& \ \|x_i\|=1}^{\text{because } C \text{ is psd}}$ & $\underbrace{\sum_i \lambda_i = 1}_{\text{because of unit trace}}$

$$\Rightarrow \mathcal{C}^* = \left\{ M^{\in S^n} \middle| \text{trace}\left(M \sum_i \lambda_i x_i x_i^T\right) \leq 1 \ \forall \right\}$$
$$= \left\{ M^{\in S^n} \middle| \sum_i \lambda_i x_i^T M x_i \leq 1 \ \forall \right\}$$

M is symm $\Rightarrow M = L\Lambda L^T$ (where Λ is diagonal & L is orthonormal)

$$\Rightarrow \mathcal{C}^* = \left\{ \overset{M=L\Lambda L^T}{\cancel{MES}} \;\middle|\; \sum_i \lambda_i\, x_i^T L \Lambda L^T x_i \le 1 \quad \forall\ \|x_i\| = 1,\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1 \right\}$$

$$= \left\{ M = L\Lambda L^T \;\middle|\; \sum_i \lambda_i \underbrace{v_i^T \Lambda v_i}_{\sigma_i} \le 1 \quad \forall\ \|v_i\| = 1,\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1 \right\}$$

($\because$ by calling $L^T x_i = v_i$)

Lets see how σ_i musts be:

$$\sup_{\lambda} \sum_i \lambda_i \sigma_i \le 1 \quad \text{s.t. } \lambda_i \geq 0,\ \sum_i \lambda_i = 1 \xrightarrow{\text{easy!}} = \max_i \sigma_i \quad \therefore \max_i v_i^T \Lambda v_i \le 1 \text{ for any } \|v_i\| = 1$$

$\Rightarrow$ each entry in $\Lambda \le 1$ ~~ext~~ else we can always choose vector $v_i \ni v_i^T \Lambda v_i > 1$.

$\therefore$ $\mathcal{C}^*$ is set of all symmetric matrices with all eigen-values ≤ 1

Let us take ~~two~~ 'n' matrices $\left.\begin{array}{l} M_1 = L \Lambda_1 L^T \\ M_2 = L \Lambda_2 L^T \\ M_n = L \Lambda_n L^T \end{array}\right\} \in \mathcal{C}^* \Rightarrow$ Λ_i will have 1 in i^{th} entry ~~$\Rightarrow \Lambda_1 \le 1$ & $\Lambda_2 \le 1$~~ above

Affine hull of them gives: $M = L \left(\sum_i s_i \Lambda_i \right) L^T$, $\sum_i s_i = 1$

which are all matrices with eigenvectors L & no restriction on eigen values

Since we started with arbitrary L, we will get all symmetric matrices as affine hull. $\therefore \dim(\mathcal{C}^*) = \dfrac{n(n+1)}{2}$.

Marking Scheme for Q2: 1 mark each for dimension questions
4 marks for coming up with eg. description.