CS 709 Problem-set

1 Theory: Convex Analysis

1.1 Domain: Hilbert Space

1. Verify whether the following are linear sets. For cases where the set is linear, provide a basis, a dual basis and determine the dimensionality:

   (a) Set of all bisymmetric matrices of size n

   Let B be set of bisymmetric matrices of size n

   $\mathrm{B} = \{\{u_{ij}\}_{i,j=1\ldots n} | \forall i,j = 1\ldots n,\ u_{ij} = u_{ji} = u_{(n-i+1)(n-j+1)} = u_{(n-j+1)(n-i+1)}\}$

   A 4×4 Bisymmetric Matrix looks like $\begin{bmatrix} a & b & c & d \\ b & e & f & c \\ c & f & e & b \\ d & c & b & a \end{bmatrix}$

   Let $\mathrm{u}, \mathrm{v} \in B$,

   $\alpha.\mathrm{u} + \mathrm{v} = \{\alpha.u_{ij} + v_{ij}\}_{i,j=1\ldots n}$

   But we have

   $u_{ij} = u_{ji} = u_{(n-i+1)(n-j+1)} = u_{(n-j+1)(n-i+1)}$ and

   $v_{ij} = v_{ji} = v_{(n-i+1)(n-j+1)} = v_{(n-j+1)(n-i+1)}$

   $\Rightarrow \alpha.u_{ij} + v_{ij} = \alpha.u_{ji} + v_{ji} = \alpha.u_{(n-i+1)(n-j+1)} + v_{(n-i+1)(n-j+1)}$

   $= \alpha.u_{(n-j+1)(n-i+1)} + v_{(n-j+1)(n-i+1)}$

   $\Rightarrow \alpha.\mathrm{u} + \mathrm{v} \in B.$

   $\Rightarrow B$ is linear set

   Basis for B will be,

$$\left\{ \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \right.$$

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \}$$

We can just decide elements of one half of lower or upper triangular matrix along other diagonal plus half of the two diagonals and rest elements will be automatically decided.

$$dim(B) = \begin{cases} n + (n-2) \dots + 3 + 1 & \text{if n is odd} \\ n + (n-2) \dots + 4 + 2 & \text{if n is even} \end{cases}$$

$$\Rightarrow dim(B) = \begin{cases} \frac{(n+1)(n+1)}{4} & \text{if n is odd} \\ \frac{n(n+2)}{4} & \text{if n is even} \end{cases}$$

(b) Set of all $n \times n$ Toeplitz matrices.

Let T be set of Topelitz matrices of size n

$\text{T} = \{\{u_{ij}\}_{i,j=1\dots n} | u_{ij} = u_{(i+1)(j+1)}\}$

A 4×4 Topelitz matrix looks like $\begin{bmatrix} a & e & f & g \\ b & a & e & f \\ c & b & a & e \\ d & c & b & a \end{bmatrix}$

Let $\text{u}, \text{v} \in T$,

$\alpha.\text{u} + \text{v} = \{\alpha.u_{ij} + v_{ij}\}_{i,j=1\dots n}$

But we have

$u_{ij} = u_{(i+1)(j+1)}, v_{ij} = v_{(i+1)(j+1)}$

$\Rightarrow \alpha.u_{ij} + v_{ij} = \alpha.u_{(i+1)(j+1)} + v_{(i+1)(j+1)}$

$\Rightarrow \alpha.\text{u} + \text{v} \in T.$

$\Rightarrow T$ is linear set

Basis for $\text{T} = \{\{u_{ij} | u_{11} = u_{22} = \dots u_{nn} = 1\},$

$\{u_{ij} | u_{12} = u_{23} = \dots = u_{(n-1)n} = 1\},$

$\{u_{ij} | u_{13} = u_{24} = \dots = u_{(n-2)n} = 1\},$

$\dots, \{u_{ij} | u_{1n} = 1\},$

$\{u_{ij} | u_{21} = u_{32} = \dots = u_{n(n-1)} = 1\},$

$\{u_{ij} | u_{31} = u_{42} = \dots = u_{n(n-2)} = 1\},$

$\dots, \{u_{ij} | u_{n1} = 1\}\}$

(Note: Rest of the entries in above matrices will be zero)

Dimension :- We can just decide elements of first row and first column and rest of the elements are automatically decided.

$\Rightarrow dim(T) = 2n - 1$

(c) Set of all $n \times n$ diagonally dominant matrices.

A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row.

It is not a linear set, as in following example it can not satisfy closure property.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A - B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

It can be easily seen that though A and B are digonally dominant matrices, A - B is not a digonally dominant matrix.

(d) Set of all doubly stochastic matrices of size n.

Doubly stachastic matrix is a square matrix of nonnegative real numbers, each of whose rows and columns sum to 1.

It can not form linear set as addition of two doubly stochastic matrices will yield non-doubly stochastic matrix (as sum of rows and columns of resultant matrix will be 2 instead of 1).

2. Show that at the end of the procedure described in lectures for reducing a spanning set to smaller sets, one would be left with a linearly independent set.

Proof:- Let $S = \{u_1, u_2, \ldots u_n\}$ be initial set of vectors.

After one by one removing vectors from S which can be represented as linear combination of earlier vectors, let we remain with set

$S' = \{u_{m_1}, u_{m_2}, \ldots u_{m_k}\}, k \leq n, m_1 =$ index of first non-zero vector in S

Assume S' is linearly dependant.

$\Rightarrow c_{m_1}.u_{m_1} + c_{m_2}.u_{m_2} + \ldots + c_{m_k}.u_{m_k} = 0$ such that $\exists c_{m_i} \neq 0$

$\Rightarrow u_{m_k} = \frac{1}{c_{m_k}}(c_{m_1}.u_{m_1} + c_{m_2}.u_{m_2} + \ldots + c_{m_{k-1}}.u_{m_{k-1}})$

But, it means u_{m_k} could have been removed. $\Rightarrow$ contradiction.

So, S' is linearly independant.

3. Show that the dimension of a subspace in a vector space is less than or equal to that of the vector space.

Proof:- Let $S \subseteq V$ be a subspace of vector space V. dim(V) = n.

Assume $dim(S) = m > n$.

Let $Basis(S) = B = \{v_1, v_2, \ldots v_m\}$

We can represent B as element of $\mathbb{R}^{m \times n}$, say A.

According to rank-nullity theorem, $rank(A) \leq ncols(A) = n < m$.

$\Rightarrow$ there must be $m - n$ linearly dependant rows in A $\Rightarrow$ contradiction.

$\Rightarrow dim(S) \leq dim(V)$

4. Show that the set of all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $\int f(x)^2$ dx is finite forms a vector space with the usual point-wise + and $\cdot$.

Proof:- Let $f, g \in \mathfrak{F}$, set of all L_2 functions.

It can be proved that $\int(\alpha.f(x) + g(x))^2$dx can be evaluated in just one way.

$\Rightarrow \int(\alpha.f(x) + g(x))^2\text{dx} = \alpha^2 \int f(x)^2\text{dx} + \int g(x)^2\text{dx} + 2\alpha \int f(x)g(x)\text{dx}$

First two terms are finite, so we have to just prove that last term is finite.

We can define inner product,

$\langle f(x), g(x)\rangle = \int f(x)g(x)\mathrm{dx}$

But $\langle f(x), g(x)\rangle \leq \|f(x)\|\|g(x)\|$

$\Rightarrow \int f(x)g(x)\mathrm{dx} \leq \int f(x)^2\mathrm{dx} \int g(x)^2\mathrm{dx}$

$\Rightarrow \int f(x)g(x) is finite$

$\Rightarrow \alpha.f(x) + g(x) \in \mathfrak{F}$

5. Consider a vector $v \in V$ in a inner-product space and a subspace $S \subset V$. Let an orthogonal basis of S be $\{v_1, \ldots, v_m\}$. Compute an expression[1] for $P_S(v)$. Show that $v - P_S(v)$ lies in the orthogonal complement of S.

   Solution:- Let $S^{'}$ be orthogonal complement of S.

   Let $Basis(S^{'}) = \{u_1, u_2, \ldots u_{n-m}\}$(as $dim(S) + dim(S^{'}) = dim(V) = n$)

   $v \in V$ can be represented as linear combination of vectors from basis of S and $S^{'}$

   $v = \lambda_1.v_1 + \lambda_2.v_2 + \ldots + \lambda_m.v_m + \lambda_1^{'}.u_1 + \lambda_2^{'}.u_2 + \ldots + \lambda_{n-m}^{'}.u_{n-m}$

   Let $x \in S$ be orthogonal projection of v on S, $P_S(v)$

   $x = \mu_1.v_1 + \mu_2.v_2 + \ldots + \mu_m.v_m$

   $v - x = (\lambda_1 - \mu_1)v_1 + (\lambda_2 - \mu_2)v_2 + \ldots + \lambda_1^{'}.u_1 + \lambda_2^{'}.u_2 + \ldots + \lambda_{n-m}^{'}.u_{n-m}$

   To minimize $\|v - x\|$, we need $\lambda_i - \mu_i = 0 \Rightarrow \mu_i = \lambda_i$

   To find out λ_i we find $\langle v, v_i\rangle$

   $\langle v, v_1\rangle = \lambda_1.\langle v_1, v_1\rangle + 0 + \ldots + 0 = \lambda_1. < v_1, v_1 >$

   (as v_1 is orthogonal to $v_i, i \neq 1$ as well as to u_j $j = 1 \ldots n - m$)

   Similarly,

   $\langle v, v_i\rangle = \lambda_i.\langle v_i, v_i\rangle$

   $\Rightarrow \lambda_i = \frac{\langle v, v_i\rangle}{\|v_i\|}$

   $\Rightarrow x = P_S(v) = \frac{\langle v, v_1\rangle}{\|v_1\|}.v_1 + \frac{\langle v, v_2\rangle}{\|v_2\|}.v_2 + \ldots + \frac{\langle v, v_m\rangle}{\|v_m\|}.v_m$

   Proof ($v - P_S(v)$ lies in the orthogonal complement of S):-

   Let $x = P_S(V)$

   $\langle v - x, v_i\rangle = \langle v, v_i\rangle - \langle x, v_i\rangle$

   $= \langle v, v_i\rangle - \frac{\langle v, v_i\rangle}{\|v_i\|}.\langle v_i, v_i\rangle$

   $= 0$

   $\Rightarrow v - P_S(V)$ is orthogonal to each $v_i, i = 1, \ldots, m$

   $\Rightarrow v - P_S(V)$ lies in orthogonal complelment of S.

6. Write down the dual/outer description for $LIN(X)$ where $X = \{[1\ 1\ 1\ 1]^\top, [1\ 1\ -1\ -1]^\top\}$. $LIN(X) \subset \mathbb{R}^4$

   Let $S^{'}$ is orthogonal complement of $LIN(X)$

[1]Expression involving the basic operations, v and $\{v_1, \ldots, v_m\}$. You should NOT use any optimization theory results other than perhaps school day knowledge about minimizing a quadratic function of a single variable.

Let $Basis(S') = \{u, v\}$ where

$\langle u_i, x_i \rangle = 0, i = 1, 2 \; x_i \in X$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow u_1 + u_2 + u_3 + u_4 = 0$ and

$u_1 + u_2 - u_3 - u_4 = 0$

Let $u_1 = 1$ and $u_2 =$ -1

$\Rightarrow u_3 = 1, u_4 = -1$ i.e. $u = [1 \; -1 \; 1 \; -1]^\top$

Similarly(by intechanging values of u_1 and u_3) , we can find $v = [-1 \; 1 \; -1 \; 1]^\top$

Dual description of $LIN(X)$:-

$LIN(X) = \{w \in V | \langle w, w' \rangle = 0, \forall w' \in X'\} \; where \; X' = \{[1 \; -1 \; 1 \; -1]^\top, [-1 \; 1 \; -1 \; 1]^\top\}$

7. Prove the following results which illustrate how limits and lin. comb.; limits and inner-products distribute. Assume $\{x_n\} \to x$, $\{y_n\} \to y$ and $\{\alpha_n\} \to \alpha$, $\{\beta_n\} \to \beta$. Here all x_n, y_n, x, y are vectors in some (finite-dim) inner-product space and all $\alpha_n, \beta_n, \alpha, \beta$ are in $\mathbb{R}$.

   (a) $\{\alpha_n x_n + \beta_n y_n\} \to \alpha x + \beta y$

   First we will prove, $\{x_n + y_n\} \to x + y$

   $\|x_n + y_n - (x + y)\| = \|(x_n - x) + (y_n - y)\|$

   $\leq \|x_n - x\| + \|y_n - y\| \ldots$ (Triangle inequality)

   $< \epsilon_1 + \epsilon_2 \ldots$ (as $\{x_n\} \to x$, $\{y_n\} \to y$)

   $\Rightarrow \|x_n + y_n - (x + y)\| < \epsilon$ where $\epsilon = \epsilon_1 + \epsilon_2$

   $\Rightarrow \{x_n + y_n\} \to x + y$

   Now, we will prove $\{\alpha_n x_n\} \to \alpha x$

   $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$

   $\leq \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\|$

   $\leq |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\|$

   $< \epsilon_1 \epsilon_2 + \epsilon_3 \|x\|$

   $< \epsilon$

   $\Rightarrow \{\alpha_n x_n\} \to \alpha x$

   Similarly we can prove,$\{\beta_n y_n\} \to \beta y$

   Combining these results we can prove the desired.

   (b) $\{\langle x_n, y_n \rangle\} \to \langle x, y \rangle$

   First, we will prove $\{\langle x_n, y \rangle\} \to \langle x, y \rangle$

   $\|\langle x_n, y \rangle - \langle x, y \rangle\|$

   $= \|\langle (x_n - x), y \rangle\|$

   $= |\langle (x_n - x), y \rangle|$

   $\leq \|(x_n - x)\|.\|y\| \ldots$ Cauchy Schwartz Inequality

   $< \epsilon_1.\|y\|$

$\Rightarrow \|\langle x_n, y\rangle - \langle x, y\rangle\| < \epsilon$ where $\epsilon = \epsilon_1.\|y\|$

$\Rightarrow \{\langle x_n, y\rangle\} \rightarrow \langle x, y\rangle$

Similarly, we can prove $\{\langle x, y_n\rangle\} \rightarrow \langle x, y\rangle$

Now, $\|\langle x_n, y_n\rangle \ - \langle x, y\rangle\| = \|\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle\|$

$= \|\langle x_n, (y_n - y)\rangle + \langle (x_n - x), y\rangle\|$

$\leq \|\langle x_n, (y_n - y)\rangle\| + \|\langle (x_n - x), y\rangle\|$

$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|$

$< \epsilon_1\epsilon_2 + \epsilon_3\|y\|$

$< \epsilon$

$\Rightarrow \{\langle x_n, y_n\rangle\} \rightarrow \langle x, y\rangle$

(c) $\{\|x_n - y_n\|\} \rightarrow \|x - y\|$
$\{\|x_n - y_n\|\} = \{\langle (x_n - y_n), (x_n - y_n)\rangle\}$

1.2 Domain: Hilbert Space

8. If S_1 and S_2 are two linear sets in a vector space $\mathcal{V}$, then show[2] that $S_1 + S_2 = LIN(S_1 \cup S_2)$.
Proof:-
Part I ($S_1 + S_2 \subseteq LIN(S_1 \cup S_2)$)
Let $u \in S_1 + S_2$
$u = u_1 + u_2,\ u_1 \in S_1, u_2 \in S_2$
$\Rightarrow u \in LIN(S_1 \cup S_2)$
$\Rightarrow S_1 + S_2 \subseteq LIN(S_1 \cup S_2)$

Part II ($LIN(S_1 \cup S_2) \subseteq S_1 + S_2$)
Let $u \in LIN(S_1 \cup S_2)$
$\Rightarrow u = \alpha_1.u_1 + \alpha_2.u_2$
But $\alpha_1.u_1 \in S_1$ and $\alpha_2.u_2 \in S_2$
$\Rightarrow u \in S_1 + S_2$
$\Rightarrow LIN(S_1 \cup S_2) \subseteq S_1 + S_2)$

9. Show that complement of an open set is closed and vice-versa [3].
Part I
Let A be open set. A^c is complement of A.
Let $\forall x_n \in A^c$, let $\{x_n\} \rightarrow x$
Assume $x \notin A^c \Rightarrow x \in A$
$\Rightarrow N_\epsilon(x) \subset A$
$\Rightarrow N_\epsilon(x) \cap A^c = \phi \Rightarrow$ contradiction
$\Rightarrow A^c$ is closed.

Part II
Let A be closed set. A^c is complement of A.
Let $x \in A^c \Rightarrow x \notin A$
$\Rightarrow x$ is not limit point of A
$\Rightarrow N_\epsilon(x) \subset A^c$
$\Rightarrow A^c$ is open.

[2]This is alternate proof of the fact that sum of two linear sets is linear.
[3]This could have been alternate definition of closed/open-ness.

10. Let $\{S_\lambda \mid \lambda \in \Lambda\}$ be a (possibly uncountable) collection of closed sets. Show that $\cap_{\lambda\in\Lambda} S_\lambda$ is a closed set[4]. Also, show that whenever the index set Λ is countable, then $\cup_{\lambda\in\Lambda} S_\lambda$ is a closed set.

Proof :-

Part I:- To prove $\bigcap_{\lambda\in\Lambda} S_\lambda$ is closed, we prove $(\bigcap_{\lambda\in\Lambda} S_\lambda)^c$ is open.

Using DeMorgan's law.

$(\bigcap_{\lambda\in\Lambda} S_\lambda)^c = \bigcup_{\lambda\in\Lambda} S^c{}_\lambda$

Let $x \in S^c{}_\lambda$ for some λ

$\Rightarrow N_\epsilon(x) \subset S^c{}_\lambda \ldots (S_\lambda \text{ is closed} \Rightarrow S^c{}_\lambda \text{ is open})$

$\Rightarrow N_\epsilon(x) \subset \bigcup_{\lambda\in\Lambda} S_\lambda{}^c$

Part II:-

[4]Through DeMorgan's laws and the above complementarity result of closed and open-ness, we get that (possibly uncountable) union of open sets is open.