# Mid-Semester Exam (CS-709)

16-Sep-2011

**Note: Please provide *rigorous and short* answers. Always *justify* why your answer may be correct. MaxMarks=20, Duration=2hrs.**

1. Consider the function $f : \mathbb{R}^{m\times n} \mapsto \mathbb{R}$ given by:

$$\begin{aligned} f(X) = \quad & \max_{u\in\mathbb{R}^m, v\in\mathbb{R}^n} e^{u^\top X v} \\ & \text{s.t.} \quad \|u\|_p \leq 1, \|v\|_q \leq 1, \end{aligned}$$

   where $p, q \in [1, \infty]$. Is $f$ convex? **[2 Marks]**

2. Let $g, h$ be convex functions defined on $\mathbb{R}^n$. Suppose $h$ is bounded below and all its level-sets are bounded. Consider $f$ defined by:

$$\begin{aligned} f(x) = \quad & \min_{y\in\mathbb{R}^n} \quad g(y) \\ & \text{s.t.} \quad h(y) \leq x, \end{aligned}$$

   whenever $x$ is such that the optimal value of the program in the RHS is finite (for other $x$, the function $f$ is not defined). Express the domain of $f$ in terms of *things related to* $g$ and/or $h$. Is the domain of $f$ (i) non-empty? (ii) bounded? (iii) convex? Is $f$ a convex function?

   **[4 Marks]**

3. Consider the function $f$ with domain as the set of all pd matrices of size $n$, with the following definition: $f(X) = trace(X^{-1})$. Is this function convex ? If so, is it strictly convex ?

   **[4 Marks]**

4. Let $f : V \mapsto \mathbb{R}$ be a closed convex function[1]. Let $f^*$ be its conjugate function. Assume that $f$ is bounded below. Note[2] that $0 \in dom(f^*)$. Is $f^*$ sub-differentiable at 0? If so, describe the sub-differential set at 0.

**[3 Marks]**

5. Let $\mathcal{P}$ be an arbitrary (given) polyhedron in $\mathbb{R}^n$. Pose the problem of finding the largest $\|\cdot\|_p$-norm ball[3] (here, $p \geq 1$) lying inside the polyhedron as a convex problem. Further, do you think this problem can be posed as a convex program with finite number of linear inequality constraints[4]?

**[3 Marks]**

6. Consider the following convex program in a finite dimensional inner-product space $\mathcal{V} = (V, +, \cdot, \langle\rangle)$:

$$\begin{array}{ll} \min\limits_{x \in V} & \langle c, x \rangle \\ \text{s.t.} & \langle a_i, x \rangle \leq b_i, \;\; \forall \; i = 1, \ldots, m \end{array}$$

Let $x^*$ be a feasible solution such that $\langle a_i, x^* \rangle = b_i, \;\; \forall \; i \in I$, where $I \subset \{1, \ldots, m\}$. Provide non-trivial[5] necessary and sufficient conditions for optimality of $x^*$. Using this characterization of the optimal solution, compute the optimal value of the above program (your expression for optimal value should not involve $x^*$).

**[4 Marks]**

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[1] The $V$ in this problem is same as that in the preceding one i.e., the set of all vectors in the space.

[2] Here, $0 \in V$ is the additive identity element in $\mathcal{V}$.

[3] Needless to say, the center of the p-norm ball need not be the origin.

[4] In this case the domain of the convex program you write needs to be a vector space. Else it is trivial to answer this question.

[5] By non-trivial condition we mean a condition which does not involve $x^*$ itself! In other words your nec.&suff. condition may involve $c, a_i, b_i, I$, but not $x^*$.

**Ans 1** $f(X) = \max_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} e^{u^T X v}$

$\underline{\text{s.t.}}$ $\|u\|_p \le 1, \ \|v\|_q \le 1$

$$= e^{\left(\begin{array}{l}\max_{u\in\mathbb{R}^n, v\in\mathbb{R}^n} u^T X v \\ \underline{\text{s.t.}} \ \|u\|_p \le 1, \|v\|_q \le 1\end{array}\right)}$$

g(x) say

( ∵ by defn. of equality of opt. probs. )

$$g(X) = \max_{u,v} u^T X v \qquad = \max_{u,v} \langle X, uv^T \rangle \longrightarrow \text{support function.}$$

$\underline{\text{s.t.}}$ $\|u\|_p \le 1, \|v\|_q \le 1$ $\qquad$ $\underline{\text{s.t.}}$ $\|u\|_p \le 1, \|v\|_q \le 1$

$$u^T X v = \text{trace}(u^T X v) = \text{trace}(v u^T X) = \langle X, uv^T \rangle$$

∴ $g(X)$ is convex (in fact conic) 1 mark

Now, $\underline{\text{TST}}$ $f(x) = e^{g(x)}$ is convex.

Total = 2 marks

Let $f(X_1) = e^{g(X_1)}$, $f(X_2) = e^{g(X_2)}$

$$\Rightarrow f(\lambda X_1 + (1-\lambda) X_2) = e^{g(\lambda X_1 + (1-\lambda) X_2)} \le e^{\lambda g(X_1) + (1-\lambda) g(X_2)}$$

( ∵ g is convex & e is monotonic increasing )

$\forall \lambda \in [0,1]$

$$\le \lambda e^{g(X_1)} + (1-\lambda) e^{g(X_2)}$$

( ∵ $e^x$ is convex + shown in lectures )

∴ $f(x)$ is convex. 1 mark

# MidSem Solutions

16/Sep/11

**Ans 2** 'h' is bounded below $\Rightarrow \min_{y\in\mathbb{R}^n} h(y) = h^* \in \mathbb{R}$

Also since 'h' is convex we have that h is continuous at all y & hence $\exists\, y^* \ni h(y^*) = h^* \leq h(y)\ \forall y$ (by Mean value theorem for example)

$$\Rightarrow \min_{y\in\mathbb{R}^n} g(y) = \infty$$
$$\underline{s.t.}\ h(y) < h^*$$

Also we get, $\min_{y\in\mathbb{R}^n} g(y) < \infty \quad \forall\, x \geq h^*$
$$\underline{\underline{s.t.}}\ h(y) \leq x$$

Now since level sets of 'h' are bounded we have the feasibility set of $\underline{s.t.}\ h(y) \leq x$ as a bounded set $\Rightarrow$ is a bounded problem!
(we proved this in lecture)

$\therefore$ For $\boxed{x \in [h^*, \infty)}$ we have that $f(x) \in \mathbb{R}$

is the domain

we know $h^* \in \mathbb{R}$ so domain is non-empty

It is not a bounded set

It is an interval hence convex

Till here (2) marks.

Easiest way to show 'f' is convex is show that its epigraph is convex:

$$\mathrm{epi}(f) = \{(x,z) \mid \underset{x\in[h^*,\infty)}{f(x) \le z}\} \quad (\because \text{defn. of epigraph})$$

$$= \{(x,z) \mid \min_{y\in\mathbb{R}^n} g(y) \le z \;\; \text{s.t. } h(y)\le x, \; x\in[h^*,\infty)\} \quad (\because \text{defn. of } f)$$

feasibility set is closed bounded convex $\therefore$ problem is always solvable!

$$= \{(x,z) \mid \exists\, y\in\mathbb{R}^n \ni g(y)\le z,\ h(y)\le x,\ x\in[h^*,\infty)\}$$

~~Let~~ We will prove convexity of epigraph using defn. itself.

Let $(x_1,z_1)\in \mathrm{epi}(f) \Rightarrow \exists\, y_1\in\mathbb{R}^n \ni g(y_1)\le z_1,\ h(y_1)\le x_1,\ x_1\ge h^*$

Let $(x_2,z_2)\in \mathrm{epi}(f) \Rightarrow \exists\, y_2\in\mathbb{R}^n \ni g(y_2)\le z_2,\ h(y_2)\le x_2,\ x_2\ge h^*$

<u>TST</u> $(\lambda x_1+(1-\lambda)x_2,\ \lambda z_1+(1-\lambda)z_2)\in \mathrm{epi}(f)$ where $\lambda\in[0,1]$

<u>Proof</u>: $g$ is convex $\Rightarrow g(\lambda y_1+(1-\lambda)y_2)\le \lambda z_1+(1-\lambda)z_2$ from this

$h$ is convex $\Rightarrow h(\lambda y_1+(1-\lambda)y_2)\le \lambda x_1+(1-\lambda)x_2$ from this

also $\lambda x_1+(1-\lambda)x_2\ge h^*$

$\Rightarrow$ the reqd. statement. Hence Proved.

2 marks

Figuring out domain & answering questions abt it gives 2 marks

Proving convexity of '$f$' gives 2 marks.

Ans 1
Consider $g(X) = \sup_{u \in \mathbb{R}^m, v \in \mathbb{R}^n} u^T X v$ $\longrightarrow = \text{trace}(u^T X v) = \text{trace}(v u^T X) = \langle X, u v^T \rangle$

s.t. $\|u\|_p \leq 1, \|v\|_q \leq 1$

$= \sup_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} \langle X, uv^T \rangle$

s.t. $\|u\|_p \leq 1, \|v\|_q \leq 1$

$=$ support function of the set $\{uv^T \mid \|u\|_p \leq 1, \|v\|_q \leq 1\}$

2 marks

So $g$ must be convex in fact it is a conic function.

Now $f(x) = e^{g(x)} \rightarrow$ convexity follows from convexity of $f(x) = e^x$

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Ans 2) $f(X) = \text{trace}(X^{-1})$

Of 5 defns. only easy way out seems to guess sub-gradient.

The guess is $\nabla f(X) = -X^{-2}$ (this is a guess from 1d case)

Now lets verify:

TST $\underbrace{\text{trace}(Y^{-1})}_{f(Y)} \geq \underbrace{\text{trace}(X^{-1})}_{f(X)} + \langle -X^{-2}, Y - X \rangle$ → sub-gradient inequality.

$\iff \text{trace}(X^{-1}) + \text{trace}(-X^{-2}(Y-X)) - \text{trace}(Y^{-1}) \leq 0$

$\iff -\text{trace}(X^{-2}Y) - \text{trace}(Y^{-1}) + 2\,\text{trace}(X^{-1}) \leq 0$

$\iff \text{trace}(X^{-2}Y + Y^{-1} - 2X^{-1}) \geq 0$

$\iff \text{trace}\left( (X^{-1}\sqrt{Y} - (\sqrt{Y})^{-1})^2 \right) \geq 0$, which is true

I

$\sqrt{Y}$ is always defined as $Y \succ 0$.

Here $f(x)$ satisfies sub-gradient inequality at all its domain points & hence is a convex function. 3 marks

From (I) we also get that:

$$\text{trace}\left(\left(X^{-1}\sqrt{Y}-(\sqrt{Y})^{-1}\right)^2\right)=0$$

$$\Longleftrightarrow \left(X^{-1}\sqrt{Y}-(\sqrt{Y})^{-1}\right)^2=0 \quad \left(\because \left(X^{-1}\sqrt{Y}-(\sqrt{Y})^{-1}\right)^2 \text{ is a psd } \cancel{\text{negative matrix}}\right)$$

trace $=0 \Rightarrow$ sum of eig $=0$ where each eig $\geq 0$

$\Rightarrow$ all eig $=0$

$\Rightarrow$ matrix is zero matrix

~~as the matrix is diagonally~~ ~~negative~~

$$\Longleftrightarrow X^{-1}\sqrt{Y}-(\sqrt{Y})^{-1}=0$$

$$\Longleftrightarrow Y=X \quad (\because \text{both } X,Y\succ 0)$$

$\therefore$ strict sub-gradient inequality is satisfied

$\therefore$ $f$ must be strictly convex. 1 mark

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~~Ans 5~~ Let the ball be ~~$B(c,r)$~~ $\{c+ru \mid \cancel{c\in\mathbb{R}^n,\ r\in\mathbb{R}},\ u\in\mathbb{R}^n,\ \|u\|_p\leq 1\}$

$\leftarrow$ no ball is determined if $c, r$ are determined

Let the given polyhedron be solution set of $a_1^T x\leq b_1$

$a_2^T x\leq b_2$

$a_m^T x\leq b_m$.

The ball is inside $\Longleftrightarrow \underbrace{a_i^T(c+ru)\leq b_i}_{}$ $\quad \forall\, i=1 \text{ to } m$, $\forall\, \|u\|_p\leq 1$

~~Each of this set $\{(c,r)\}$~~

The problem can be posed as:

$$\max_{c \in \mathbb{R}^n, r \in \mathbb{R}} \quad r$$

$$\underline{\underline{\text{s.t.}}} \quad a_i^T(c + ru) \leq b_i \quad \forall\, i = 1 \text{ to } m,\ \|u\|_p \leq 1,$$
$$r \geq 0$$

$$= - \boxed{\begin{aligned} &\min_{c \in \mathbb{R}^n, r \in \mathbb{R}} -r \\ &\underline{\underline{\text{s.t.}}} \quad a_i^T(c + ru) \leq b_i \quad \forall\, i = 1 \text{ to } m,\ \|u\|_p \leq 1, \\ &\qquad r \geq 0 \end{aligned}}$$

I

1 mark

we will show this is a convex program.

domain of I: $\{(c, r) \mid c \in \mathbb{R}^n, r \in \mathbb{R}\} \longrightarrow$ obviously convex.

Objective I: $f(c, r) = -r \longrightarrow$ linear function hence convex.

Constraint set of I $= \mathcal{C} = \bigcap_{\substack{i = 1 \text{ to } m, \\ \{u / \|u\|_p \leq 1\}}} C_{iu}$

where $C_{iu} = \{(c, r) \mid a_i^T c + (a_i^T u) r \leq b_i\} \longrightarrow$ which is an affine set $\therefore$ convex set.

$\mathcal{C}$ is uncountable intersection of $C_{iu}$ (convex) over all $i = 1$ to $m$ & $\{u / \|u\|_p \leq 1\}$

Hence $\mathcal{C}$ is convex. $\therefore$ I is a convex program.

1 mark

Now, $\bigcap_{\{u / \|u\|_p \le 1\}} \mathcal{C}_{iu} = \left\{ (c, r) \,/\, a_i^T c + (a_i^T u) r \le b_i \;\; \forall \|u\|_p \le 1 \right\}$

$$= \left\{ (c, r) \,/\, a_i^T c + \left( \sup_{\|u\|_p \le 1} a_i^T u \right) r \le b_i \right\}$$

Let $D_i \equiv$

$$= \left\{ (c, r) \,/\, a_i^T c + \|a_i\|_q r \le b_i \right\}$$

$\frac{1}{p} + \frac{1}{q} = 1$

this is again an affine set!

$\therefore \mathcal{C} = \bigcap_{i=1}^{m} D_i$

$\therefore$ (I) is same as:

$$- \boxed{\begin{aligned} &\min_{c \in \mathbb{R}^n, r \in \mathbb{R}} \quad -r \\ &\underline{\underline{\text{s.t.}}} \quad a_1^T c + \|a_1\|_q r \le b_1, \\ &\qquad\quad \vdots \\ &\qquad\quad a_m^T c + \|a_m\|_q r \le b_m \end{aligned}}$$

This is infact example of a Linear Program

½ mark

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(6) We know for a CP Convex Program $\min_{x \in X} f(x)$ s.t. $x \in C$, $X \cap C = \mathcal{F}$

$x^*$ is optimal solution $\iff \langle \nabla f(x^*), T_{\mathcal{F}}(x^*) \rangle \ge 0$

tgt. cone

$\iff \nabla f(x^*) \in N_{\mathcal{F}}(x^*)$

normal cone

(I)

Now, $f(x) = \langle c, x \rangle$

$$\Rightarrow \boxed{\nabla f(x^*) = c} \quad \text{II}$$

Now we need to determine $T_{\mathcal{F}}(x^*) = \{h \mid \exists t > 0 \ni x^* + th \in \mathcal{F}\}$

We know for $i \in \{1, \dots, m\} \cap I^c$ we have $\langle a_i, x^* \rangle < b_i$

so $h$ can be any direction.

So $i \in I$ only determine the tangent cone.

$\rightarrow$ we have $\langle a_i, x^* \rangle = b_i$

~~we know~~ Since $\mathcal{F}$ is a polyhedron & we know:

$$T_{\mathcal{F}}(x^*) = \{x - x^* \mid x \text{ satisfies}\}$$

$$= \left\{ x - x^* \;\middle|\; \begin{array}{l} \langle a_i, x^* \rangle = b_i, \\ \langle a_i, x \rangle \le b_i \\ \forall i \in I \end{array} \right\}$$

$$= \{x - x^* \mid \langle a_i, x - x^* \rangle \le 0 \ \forall i \in I\}$$

$\therefore$ dual description of $T_{\mathcal{F}}(x^*)$ is $\{a_i \mid i \in I\}$

$\therefore$ inner description of its dual cone i.e. $N_{\mathcal{F}}(x^*) = \{-a_i \mid i \in I\}$ III

(we showed this in lectures)

3 marks

$\therefore$ by I, II, III we have:

$$\boxed{x^* \text{ is optimal soln.} \iff -c \in \text{conichull}(\{-a_i \mid i \in I\})}$$

$$\exists \lambda_i \geq 0 \ni$$
$$\Leftrightarrow \quad c = \sum_{i \in I} \lambda_i (a_i)$$

Now optimal value $= \langle c, x^* \rangle$

$$= \langle -\sum_{i \in I} \lambda_i a_i, x^* \rangle$$

$$= -\sum_{i \in I} \lambda_i \langle a_i, x^* \rangle$$

$$= -\sum_{i \in I} \lambda_i b_i$$ 1 mark

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Ans 4 $\quad f^*(y) = \sup_{x \in V} \langle x, y \rangle - f(x) \quad$ (I)

This itself is the dual description of $f^*$ $\rightarrow$ So we must be able to get the sub-gradient using this expression.

$f^*(0) = -\inf_{x \in V} f(x) = \underline{f} \in \mathbb{R}$ (as $f$ is bounded below)

$\therefore 0 \in \operatorname{dom}(f^*)$

Again by similar argument as in Ans 2 $\exists x^* \ni f(x^*) \leq f(x) \; \forall x \in V$, $f(x^*) = \underline{f}$; infact $x^* \in \operatorname{argmin}_{x \in V} f(x)$

$\Rightarrow -f^*(0) = f(x^*) \leq f(x) \quad \forall x \in V$

From (I) we have $f^*(y) \geq \langle x^*, y \rangle - f(x^*)$

$$\geq f^*(0) + \langle x^*, y - 0 \rangle$$

So we get that:

Any $x^* \in \underset{x \in V}{\operatorname{argmin}} f(x)$ satisfies subgradient inequality at 0 for $f^*$

$$\Rightarrow \boxed{\partial f^*(0) = \underset{x \in V}{\operatorname{argmin}} f(x)}$$

3 marks