End-Semester Exam (CS-709)

24-Nov-2011

Note: Please provide *rigorous and short* answers. Always *justify* why your answer may be correct. MaxMarks=30, Duration=3hrs.

1. Let $S_C(x)$ represent the support function of the set $C \subset \mathbb{R}^n$ evaluated at the point $x \in \mathbb{R}^n$. Then, is the following equality:

$$\begin{aligned} S_{C_1 \cap C_2}(x) = \min_{x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^n} & \quad S_{C_1}(x_1) + S_{C_2}(x_2), \\ \text{s.t.} & \quad x_1 + x_2 = x, \end{aligned}$$

true for any $C_1, C_2 \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$? If not, can you write down under what conditions on C_1, C_2 will the equality be true for all $x \in \mathbb{R}^n$?

[4 Marks]

2. Consider the following optimization problem:

$$\begin{aligned} \min_{a \in \mathbb{R}^n, b \in \mathbb{R}} & \quad \|a\|_2, \\ \text{s.t.} & \quad a^\top x_i - b \geq 1 \; \forall \; i = 1, \ldots, m_x, \\ & \quad a^\top y_i - b \leq -1 \; \forall i = 1, \ldots, m_y, \end{aligned} \tag{1}$$

where $x_1, \ldots, x_{m_x}$ and $y_1, \ldots, y_{m_y}$ are given points in $\mathbb{R}^n$. Geometrically, the problem above is that of finding a hyperplane $a^\top x - b = 0$ with least $\|a\|_2$ such that the set of points in $X = \{x_1, \ldots, x_{m_x}\}$ are constrained to be strictly in the positive half space and the set of points in $Y = \{y_1, \ldots, y_{m_y}\}$ are constraiend to be strictly in the negative half space of the hyperplane.

Now, in real-world applications the points in X, Y may be obtained from some simulations/experiments. Hence they may not be known exactly. What is more practical is to assume that a region in which each point x_i/y_i most probably lies is available i.e., instead of x_i we assume that a set $R_i \subset \mathbb{R}^n$ which has a high probability of x_i lying in is given. Similarly assume that instead of y_i, we are given $S_i \subset \mathbb{R}^n$. Ofcourse, now the only change from (1) will be that we would like to constrain the hyperplane such that for all possible locations of x_i and y_i, the strict separation happens. This can be expressed as the following optimization problem:

$$\begin{aligned} \min_{a \in \mathbb{R}^n, b \in \mathbb{R}} \quad & \|a\|_2, \\ \text{s.t.} \quad & a^\top x_i - b \geq 1 \ \forall \ x_i \in R_i, \ i = 1, \ldots, m_x, \\ & a^\top y_i - b \leq -1 \ \forall y_i \in S_i, \ i = 1, \ldots, m_y, \end{aligned} \tag{2}$$

Usually, (2) is refered to as the robust variant of (1), as the latter is robust to the uncertainities in the positions of the points to be separated.

Depending on the application and the nature of simulations, the following shapes for uncertainity regions R_i, S_i are common:

Point: $R_i = \{x_i\}, S_i = \{y_i\}$. In this case the points are known exactly and (2) is equal to (1).

Box: $R_i = \{x \mid \|\Lambda_i(x - x_i)\|_\infty \leq 1\}$, where Λ_i is a diagonal matrix with positive entries. Similarly, $S_i = \{y \mid \|\Sigma_i(y - y_i)\|_\infty \leq 1\}$, where Σ_i is a diagonal matrix with positive entries.

Ellipse: $R_i = \{x \mid \|\bar{\Lambda}_i(x - x_i)\|_2 \leq 1\}$, where $\bar{\Lambda}_i$ is a pd matrix. Similarly, $S_i = \{y \mid \|\bar{\Sigma}_i(y - y_i)\|_2 \leq 1\}$, where $\bar{\Sigma}_i$ is a pd matrix.

Box-Ellipse: $R_i = \{x \mid \|\Lambda_i(x - x_i)\|_\infty \leq 1, \ \|\bar{\Lambda}_i(x - x_i)\|_2 \leq 1\}$, where Λ_i is a diagonal matrix with positive entries and $\bar{\Lambda}_i$ is a pd matrix. Similarly, $S_i = \{y \mid \|\Sigma_i(y - y_i)\|_\infty \leq 1, \ \|\bar{\Sigma}_i(y - y_i)\|_2 \leq 1\}$, where Σ_i is a diagonal matrix with positive entries and $\bar{\Sigma}_i$ is a pd matrix.

With each of the above four uncertainty regions, re-write the problem (2) as an SOCP. After doing this, consider the special case where all $\bar{\Lambda}_i$ and $\bar{\Sigma}_i$ are additionally restrcited to be diagonal. Now, with each of the four uncertainty regions, re-write the problem (2) as an SOCP with a **single** non-linear SOC constraint and some linear/affine inequality constraints[1].

Ofcourse, an SOCP with a single (non-linear) SOC is far more special than a generic SOCP. So it may not be wise to give the problem to cvx, as it will use a generic SOCP solver which *may not* exploit this speciality. Now, using your knowledge about optimization algorithms taught in lectures and encountered in projects, can you provide an algorithm which will exploit this speciality and solve the problem *efficiently*? You may also think[2] of posing the SOCP with single non-linear SOC as another problem to do this.

3.

[1] Note that linear/affine inequalities are also SOCs. Hence when we write "non-linear SOC", we mean an SOC which is not a linear/affine inequality

[2] This is kind of an open-question for your background. So, I am not asking for the best algorithm, but an algorithm which exploits this speciality and hence be able to solve problems in high dimension or high m_x and/or m_y efficiently.