IPL-10. Assignment 1. Solutions

Q1. When a needle is dropped, it has three parameters, which which we can assume to be independent: horizontal position (x), vertical position (y) and orientation (θ). Since the grid is infinitely repeated, we can assume x and y taking values within any single square. θ takes values from 0 to π (due to symmetry, we can consider for the case till $\pi/2$ only). For a given θ, let's calculate the probability that the needle doesn't cross the square, and average this for all values of θ. So from the figure below, the needle doesn't intersect

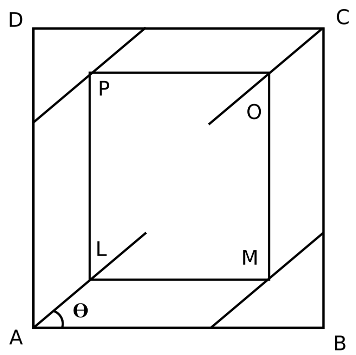


Figure 1: Region of centre of needle where it doesn't intersect.

when its centre lies within the square $LMOP$. Hence the probability is given by,

$$\frac{\text{Area LMOP}}{\text{Area ABCD}} = \frac{(l - \frac{l}{2}\cos\theta)(l - \frac{l}{2}\sin\theta)}{l^2}$$

Averaging for θ taking values from 0 to $\pi/2$,

$$\frac{\pi}{2}\int_0^{\pi/2}(1 - \frac{1}{2}\sin\theta - \frac{1}{2}\cos\theta + \frac{1}{4}\sin\theta\cos\theta)\mathrm{d}\theta,$$

which evaluates to $1 - \frac{7}{4\pi}$. So the probability that the needle does intersect the grid is $\frac{7}{4\pi}$.

Q2. $\mathcal{F}$ is the set of intervals of the type $(a, b]$ and their finite unions, i.e. $\cup_{i=1}^n (a_i, b_i]$.

Now for an element of type $(a, b]$, its complement is $(-\infty, a] \cup (b, \infty)$. But $(-\infty, a] \in \mathcal{F}$ and $(b, \infty) \in \mathcal{F}$, therefore, so is $(-\infty, a] \cup (b, \infty)$.

Now for an element of type $\cup_{i=1}^n (a_i, b_i]$, its complement is

$$\begin{aligned}(\cup_{i=1}^n (a_i, b_i])^c &= \cap_{i=1}^n (a_i, b_i]^c \qquad \text{[DeMorgan's Law]} \\ &= \{\ x \mid x \leq a_i \text{ or } x > b_i, \forall i = 1, \ldots, n\} \\ &= \{\ x \mid x \leq a \text{ or } x > b, a = \min_i a_i, b = \min_i b_i\} \\ &= (a, b]^c \in \mathcal{F}\end{aligned}$$

Hence, $\mathcal{F}$ is closed under complementation. Although, $\mathcal{F}$ is not closed under union. Consider the countable union: $\cup_{i=1}^{\infty}(a, b - \frac{1}{i}] = (a,b) \notin \mathcal{F}$. Hence, $\mathcal{F}$ is not a σ-algebra.

Q3.i. To show

$$P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{1 \le i < j \le n} P(E_i \cap E_j) + \sum_{1 \le i < j < k \le n} P(E_i \cap E_j \cap E_k) + \ldots + (-1)^n \sum_{i=i}^{n} P(\cap_{j=1, j\neq i}^{n} E_i) + (-1)^{n+1} P(\cap_{i=1}^{n} E_i).$$

Base case: for $n = 2$, it holds. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \sum_{i=1}^{2} E_i + (-1)^3 P(\cap_{i=1}^{2} E_i)$.

Inductive step: assume it holds for n. We show that it also holds for $n+1$.

$$P(\cup_{i=1}^{n+1} E_i) = P((\cup_{i=1}^{n} E_i) \cup E_{n+1}) \quad (1)$$
$$= P(\cup_{i=1}^{n} E_i) + P(E_{n+1}) - P((\cup_{i=1}^{n} E_i) \cap E_{n+1}) \quad (2)$$

Now,

$$P((\cup_{i=1}^{n} E_i) \cap E_{n+1})$$
$$= P(\cup_{i=1}^{n}(E_i \cap E_{n+1})) \quad \text{[distributive law]}$$
$$= \sum_{i=1}^{n} P(E_i \cap E_{n+1}) - \sum_{1 \le i < j \le n} P((E_i \cap E_{n+1}) \cap (E_j \cap E_{n+1}))$$
$$+ \sum_{1 \le i < j < k \le n} P((E_i \cap E_{n+1}) \cap (E_j \cap E_{n+1}) \cap (E_k \cap E_{n+1})) +$$
$$\ldots + (-1)^n \sum_{i=i}^{n} P(\cap_{j=1, j\neq i}^{n}(E_i \cap E_{n+1})) + (-1)^{n+1} P(\cap_{i=1}^{n}(E_i \cap E_{n+1}))$$

Observe now that, substituting in equation 2, these are exactly the terms required to get the E_{n+1} set included and having the equation in required form.

Q3.ii. To prove Boole's inequality:

$$P(\cup_{i=1}^{n} E_i) \le \sum_{i=1}^{n} P(E_i)$$

Base case: for $n = 1$, we have $P(E_1) \le P(E_1)$.

Inductive step: Assume the equation holds for n. We show that it also holds for $n+1$.

$$P(\cup_{i=1}^{n+1} E_i) = P((\cup_{i=1}^{n} E_i) \cup E_{n+1})$$
$$= P(\cup_{i=1}^{n} E_i) + P(E_{n+1}) - P((\cup_{i=1}^{n} E_i) \cap E_{n+1})$$
$$\le P(\cup_{i=1}^{n} E_i) + P(E_{n+1})$$
$$\le \sum_{i=1}^{n} P(E_i) + P(E_{n+1})$$
$$= \sum_{i=1}^{n+1} P(E_i).$$

Now to prove (left side of the inequalities)

$$\sum_{i=1}^{n} P(E_i) - \sum_{1\le i<j\le n} P(E_i \cap E_j) \le P(\cup_{i=1}^{n} E_i)$$

Again, the base case is easy to check. To prove the inductive part (for $n+1$):

$$\begin{aligned}
&\sum_{i=1}^{n+1} P(E_i) - \sum_{1\le i<j\le n+1} P(E_i \cap E_j) \\
&= \sum_{i=1}^{n} P(E_i) + P(E_{n+1}) - \sum_{1\le i<j\le n} P(E_i \cap E_j) - \sum_{i=1}^{n} P(E_i \cap E_{n+1}) \\
&\le P(\cup_{i=1}^{n} E_i) + P(E_{n+1}) - \sum_{i=1}^{n} P(E_i \cap E_{n+1}) \\
&= P((\cup_{i=1}^{n} E_i) \cup E_{n+1}) - P((\cup_{i=1}^{n} E_i) \cap E_{n+1}) - \sum_{i=1}^{n} P(E_i \cap E_{n+1}) \\
&\le P((\cup_{i=1}^{n} E_i) \cup E_{n+1}) = P(\cup_{i=1}^{n+1} E_i).
\end{aligned}$$

Q3.iii. To prove

$$P(\cap_{i=1}^{n} E_i) \ge \sum_{i=1}^{n} P(E_i) - n + 1$$

For base case it's easy to check it holds. For the inductive step:

$$\begin{aligned}
P(\cap_{i=1}^{n+1} E_i) &= P((\cap_{i=1}^{n} E_i) \cap E_{n+1}) \\
&= P(\cap_{i=1}^{n} E_i) + P(E_{n+1}) - P((\cap_{i=1}^{n} E_i) \cup E_{n+1}) \\
&\ge \sum_{i=1}^{n} P(E_i) - n + 1 + P(E_{n+1}) - P((\cap_{i=1}^{n} E_i) \cup E_{n+1}) \\
&= \sum_{i=1}^{n+1} P(E_i) - n + 1 - P((\cap_{i=1}^{n} E_i) \cup E_{n+1}) \\
&\ge \sum_{i=1}^{n+1} P(E_i) - n + 1 - 1 \\
&= \sum_{i=1}^{n+1} P(E_i) - (n+1) + 1.
\end{aligned}$$

(Note that in the proofs above, the inequalities are simplified by using the properties of probabilities: $P(E) \ge 0$ and $P(E) \le 1$.)

Q4. X is a valid RV if intervals of the type $(-\infty, x], \forall x \in \mathbb{R}$ have corresponding valid events in $\mathcal{F}$. This is, $X^{-1}((-\infty, x]) \in \mathcal{F}$. So for the given RV,

$$X^{-1}((-\infty, x]) = \{\ \omega \mid X(\omega) \in (-\infty, x]\}$$
$$= \begin{cases} \Omega = [0,1] & x \geq 1 \\ [0, x] & 0 \leq x < 1 \\ \emptyset & x < 0 \end{cases}$$

Hence events of the type $[0, x], \forall x \in [0, 1]$ must be in $\mathcal{F}$ for X to be a valid RV.

Now let us compute the distribution functions of X and Y.

$$F_X(x) = P[X \leq x] = P(\{\omega \in \Omega \mid X(\omega) \leq x\})$$
$$= P(\{\omega \in \Omega \mid \omega \leq x\})$$
$$= \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Similarly,

$$F_Y(y) = P[Y \leq y] = P(\{\omega \in \Omega \mid Y(\omega) \leq y\})$$
$$= P(\{\omega \in \Omega \mid 1 - \omega \leq y\})$$
$$= \begin{cases} 0 & y < 0 \\ 1 - (1 - y) = y & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

Q5. To show a discrete RV X is geometric iff $P[X > m+n|X > n] = P[X > m]$.
($\Rightarrow$): For a geometric RV, we know that $P[X \leq k] = 1 - (1 - p)^k$. So $P[X > k] = (1 - p)^k$.

$$P[X > m + n|X > n] = \frac{P[X > m + n, X > n]}{P[X > n]}$$
$$= \frac{P[X > m + n]}{P[X > n]}$$
$$= \frac{(1 - p)^{m+n}}{(1 - p)^n}$$
$$= (1 - p)^m = P[X > m].$$

($\Leftarrow$):

$$P[X > m + n|X > n] = P[X > m]$$
$$\frac{P[X > m + n, X > n]}{P[X > n]} = P[X > m]$$
$$P[X > m + n] = P[X > m].P[X > n]$$

Put $n = 1$, we get

$$\begin{aligned} P[X > m+1] &= P[X > m].P[X > 1] \\ \Rightarrow P[X > m] &= P[X > m-1].P[X > 1] \\ &\dots \\ P[X > 2] &= P[X > 1].P[X > 1] \end{aligned}$$

Substituting the value m times we get $P[X > m+1] = (P[X > 1])^{m+1}$.
But $P[X > 1] = 1 - P[X = 1] = 1 - p$. So,

$$\begin{aligned} P[X = m] &= P[X \leq m] - P[X \leq m-1] \\ &= 1 - (1-p)^m - (1 - (1-p)^{m-1}) \\ &= (1-p)^{m-1}p. \end{aligned}$$

This is the pmf of geometric RV.

Q8. We can choose the set A in 2^n ways. Similarly, B can be chosen in 2^n ways. So the size of sample space is $2^n \cdot 2^n = 2^{2n}$. Now there are ${}^nC_i$ ways to choose the set B of size i, and for each such selected B of size i, there are 2^i ways of selecting A such that $A \subseteq B$. Summing this for all values of i gives us the number of ways in which A will be a subset of B. Hence,

$$P(A \subseteq B) = \frac{\sum_{i=0}^{n} 2^i \cdot {}^nC_i}{2^{2n}}$$

Similarly, for a selected set B of size i, we can select A in 2^{n-i} ways from its complement. Hence,

$$P(A \cap B = \emptyset) = \frac{\sum_{i=0}^{n} 2^{n-i} \cdot {}^nC_i}{2^{2n}}$$

IPL-10. Assignment 2. Solutions

Q9. Given X is standard normal and $Y = X^2$.

$$\begin{aligned} F_Y(y) &= P[Y \le y] \\ &= P[X^2 \le y] \\ &= \begin{cases} 0 & y \le 0 \\ P[-\sqrt{y} \le X \le \sqrt{y}] & y > 0 \end{cases} \\ &= \begin{cases} 0 & y \le 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y > 0 \end{cases} \end{aligned}$$

Now,

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y} \\ &= \begin{cases} 0 & y \le 0 \\ \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) & y > 0 \end{cases} \\ &= \begin{cases} 0 & y \le 0 \\ \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} & y > 0 \quad [\text{Since } f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}] \end{cases} \end{aligned}$$

Now let us find the pdf of $Z = \sqrt{Y}$. Note that the range of Z is $(0, \infty)$. Also, Z is monotonic, differentiable function of RV. Hence,

$$\begin{aligned} f_Z(z) &= f_Y(g^{-1}(z)) \left| \frac{d}{dz} g^{-1}(z) \right| \\ &= \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi z^2}} \left| \frac{d}{dz} z^2 \right| \qquad [\text{Here } g(y) = \sqrt{y},\ g^{-1}(z) = z^2] \\ &= \sqrt{\tfrac{2}{\pi}}\, e^{-\frac{z^2}{2}} \qquad \forall z > 0. \end{aligned}$$

Q10. We know that $r = \frac{v^2}{g} \sin 2\theta$, where r is the range of the projectile, v is the initial velocity, g is acceleration due to gravity and θ is the projection angle. Θ is uniform RV between $[0, \pi/2]$. Hence, range $R = \frac{v^2}{g} \sin 2\Theta$.

$$\begin{aligned} F_R(r) &= P[R \le r] \\ &= P[\tfrac{v^2}{g} sin 2\Theta \le r] \\ &= \begin{cases} 0 & r < 0 \\ P[0 \le \Theta \le \frac{1}{2}\sin^{-1}\left(\frac{rg}{v^2}\right)] + P[\frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{rg}{v^2}\right) \le \Theta \le \frac{\pi}{2}] & 0 \le r < \frac{v^2}{g} \\ 1 & r \ge \frac{v^2}{g} \end{cases} \\ &= \begin{cases} 0 & r < 0 \\ \frac{2}{\pi}\sin^{-1}\left(\frac{rg}{v^2}\right) & 0 \le r < \frac{v^2}{g} \\ 1 & r \ge \frac{v^2}{g} \end{cases} \end{aligned}$$

Hence we have,

$$
\begin{aligned}
f_R(r) &= \frac{\mathrm{d}F_R(r)}{\mathrm{d}r} \\
&= \begin{cases} \frac{2}{\pi}\frac{1}{\sqrt{(v^2/g)^2 - r^2}} & 0 \le r < \frac{v^2}{g} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Q11. This is a solved example in Papoulis and Pillai, page 148.

Q12. We will have to prove this for X being a discrete as well as continuous RV.
For the discrete case:

$$
\begin{aligned}
\mathbb{E}[|X - c|] &= \sum_{x \in E} |x - c| f_X(x) \\
&= \sum_{x \in E, x \le c} (c - x) f_X(x) + \sum_{x \in E, x > c} (x - c) f_X(x) \\
&= g(c) \text{ (say)}
\end{aligned}
$$

Now $\mathbb{E}[|X - c|]$ a function of c. A necessary condition for c^* minimizing g is $g'(c^*) = 0$.

$$
\begin{aligned}
g'(c) &= \sum_{x \in E, x \le c} f_X(x) - \sum_{x \in E, x > c} f_X(x) \qquad (1) \\
&= F_X(c) - (1 - F_X(c)) \\
&= 2F_X(c) - 1
\end{aligned}
$$

Hence we have $F_X(c^*) = \frac{1}{2}$, which means c^* is the median (by definition). We also need to verify that this is minima. This follows from equation 1, where for $c < c^*, g'(c) \le 0$ while for $c > c^*, g'(c) \ge 0$. Hence c^* must give a minimum.

Now for the continuous case. Similar to the previous case, let

$$
\begin{aligned}
g(c) &= \mathbb{E}[|X - c|] \\
&= \int_{-\infty}^{\infty} |x - c| f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{c} (c - x) f_X(x)\mathrm{d}x + \int_{c}^{\infty} (x - c) f_X(x)\mathrm{d}x \\
&= cF_X(c) - c(1 - F_X(c)) + \int_{c}^{\infty} x f_X(x)\mathrm{d}x - \int_{-\infty}^{c} x f_X(x)\mathrm{d}x \\
&= 2cF_X(c) - c + \int_{c}^{\infty} x f_X(x)\mathrm{d}x - \int_{-\infty}^{c} x f_X(x)\mathrm{d}x
\end{aligned}
$$

Now

$$
\begin{aligned}
g'(c) &= 2F_X(c) + 2cf_X(c) - 1 + \frac{d}{dc}\left[\int_{c}^{\infty} x f_X(x)\mathrm{d}x\right] - \frac{d}{dc}\left[\int_{-\infty}^{c} x f_X(x)\mathrm{d}x\right] \\
&= 2F_X(c) + 2cf_X(c) - 1 + (-cf_X(c)) - (cf_X(c)) \\
&= 2F_X(c) - 1
\end{aligned}
$$

Again, necessary condition for c^* to be minimizer is $g'(c^*) = 0 \Rightarrow F_X(c^*) = \frac{1}{2}$. Also, $g''(c) = 2f_X(c) \geq 0 \; \forall c$. Hence c^* is minimizer.

Q13. We use the inequality $\log a \leq a - 1$ to prove this. Here, let $a = f_Y(x)/f_X(x)$. Hence we have

$$\log f_Y(x) - \log f_X(x) = \log \frac{f_Y(x)}{f_X(x)} \leq \frac{f_Y(x)}{f_X(x)} - 1$$

Multiplying by $f_X(x)$ and integrating, we get

$$\int_{-\infty}^{\infty} [\log f_Y(x) - \log f_X(x)] f_X(x) \mathrm{d}x \leq \int_{-\infty}^{\infty} [f_Y(x) - f_X(x)] \mathrm{d}x$$
$$\Rightarrow \quad \mathbb{E}[\log f_Y(X)] - \mathbb{E}[\log f_X(X)] \leq 0$$

RHS is 0 since f_X and f_Y are density functions.

Q14. Probability that a blue ball is picked is $\frac{m_1}{m}$ (call it p). Probability of red ball is picked is $\frac{m_2}{m} = (1-p)$. Let's calculate the pmf of X. At least two balls will be drawn.

$$\begin{aligned}
P[X = 2] &= p(1-p) + (1-p)p \\
P[X = 3] &= p^2(1-p) + (1-p)^2 p \\
&\vdots \\
P[X = n] &= p^{n-1}(1-p) + (1-p)^{n-1} p
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n\geq 2} n(p^{n-1}(1-p) + (1-p)^{n-1}p) \\
&= (1-p)\sum_{n\geq 2} np^{n-1} + p\sum_{n\geq 2} n(1-p)^{n-1} \\
&= (1-p)\sum_{n\geq 1} (n+1)p^n + p\sum_{n\geq 1}(n+1)(1-p)^n \\
&= (1-p)\sum_{n\geq 0} np^n + (1-p)\left(\sum_{n\geq 0} p^n - 1\right) + p\sum_{n\geq 0} n(1-p)^n \\
&+ p\left(\sum_{n\geq 0}(1-p)^n - 1\right) \\
&= (1-p)\frac{p}{(1-p)^2} + (1-p)\left(\frac{1}{1-p} - 1\right) + p\frac{1-p}{p^2} + p\left(\frac{1}{p} - 1\right) \\
&= \frac{p}{1-p} + \frac{1}{p}
\end{aligned}$$

Note, above we've used the identities $\sum_{n\geq 0} p^n = 1/(1-p)$ and $\sum_{n\geq 0} np^n = p/(1-p)^2$.

Now for the case of without replacement. Wlog, assume that $m_1 \leq m_2$.

$$
\begin{aligned}
P[X=2] &= \tfrac{m_1}{m}\tfrac{m_2}{m-1} + \tfrac{m_2}{m}\tfrac{m_1}{m-1} \\
P[X=3] &= \tfrac{m_1}{m}\tfrac{m_1-1}{m-1}\tfrac{m_2}{m-2} + \tfrac{m_2}{m}\tfrac{m_2-1}{m-1}\tfrac{m_1}{m-2} \\
&\ldots \text{ for } n \leq m_1+1, \\
P[X=n] &= \tfrac{m_1}{m}\tfrac{m_1-1}{m-1}\cdots\tfrac{m_1-n+2}{m-n+2}\tfrac{m_2}{m-n+1} + \tfrac{m_2}{m}\tfrac{m_2-1}{m-1}\cdots\tfrac{m_2-n+2}{m-n+2}\tfrac{m_1}{m-n+1} \\
&= \tfrac{m_1!/(m_1-n+1)!\,m_2}{m!/(m-n)!} + \tfrac{m_2!/(m_2-n+1)!\,m_1}{m!/(m-n)!} \\
&\text{for } m_1+1 < n \leq m_2+1, \\
P[X=n] &= \tfrac{m_2!/(m_2-n+1)!\,m_1}{m!/(m-n)!}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}[X] = &\sum_{n \leq m_1+1} n \left(\tfrac{m_1!/(m_1-n+1)!\,m_2}{m!/(m-n)!} + \tfrac{m_2!/(m_2-n+1)!\,m_1}{m!/(m-n)!} \right) \\
&+ \sum_{m_1+1<n\leq m_2+1} n \left(\tfrac{m_2!/(m_2-n+1)!\,m_1}{m!/(m-n)!} \right)
\end{aligned}
$$

(Can this be simplified?)

(15) $P[X \geq \varepsilon] \leq P[e^{sX} \geq e^{s\varepsilon}]$ $(\forall s > 0)$

$$\leq \frac{E\{e^{sX}\}}{e^{s\varepsilon}} \quad \forall s > 0$$

$$= e^{s^2/2 - s\varepsilon} \quad \forall s > 0$$

$$\Rightarrow P[X \geq \varepsilon] \leq e^{\min_{s>0} s^2/2 - s\varepsilon}$$

$$= e^{-\varepsilon^2/2} \quad (\text{assuming } \varepsilon > 0)$$

Now if $\varepsilon < 0$ then let $-\varepsilon = \varsigma > 0$

$$P[X \ngeq \varepsilon] = 1 - P[X \leq \varepsilon] = 1 - P[-X \geq \varsigma]$$

Now distribution of X & $-X$ are the same (why??)

So, $P[-X \geq \varsigma] \leq e^{-\varsigma^2/2}$

$$\Rightarrow P[X \geq \varepsilon] \geq 1 - e^{-\varepsilon^2/2}$$

[If somebody wrongly proved the inequality $\forall \varepsilon$ irrespective of its sign, then marks should be appropriately cut]

IPL-10. Assignment 2. Solutions

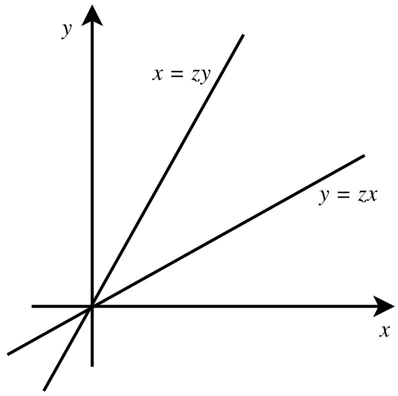
Q16. $Z = \frac{min(X,Y)}{max(X,Y)}$. This function can be written as

$$Z = \begin{cases} X/Y & X \leq Y \\ Y/X & X > Y. \end{cases}$$

Hence

$$\begin{aligned} F_Z(z) &= P[X/Y \leq z, X \leq Y] + P[Y/X \leq z, X > Y] \\ &= P[X \leq zY, X \leq Y] + P[Y \leq zX, X > Y]. \end{aligned}$$

Since X and Y take only positive values, $0 \leq Z \leq 1$. From the figure, our



favourable region is the one below the line $y = zx$ and to the left of the line $x = zy$. Hence,

$$F_Z(z) = \int_0^\infty \int_{x=0}^{zy} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_{y=0}^{zx} f_{XY}(x,y)\,dx\,dy$$

Differentiating wrt z,

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(yz,y)\,dy + \int_0^\infty x f_{XY}(x,xz)\,dx \\ &= \int_0^\infty y(f_{XY}(yz,y) + f_{XY}(y,yz))\,dy \end{aligned}$$

When X and Y are iid exponential,

$$\begin{aligned} f_Z(z) &= \int_0^\infty y\lambda^2(e^{-\lambda(yz+y)} + e^{-\lambda(y+yz)})\,dy \\ &= 2\lambda^2 \int_0^\infty y e^{-\lambda y(1+z)}\,dy \\ &= \begin{cases} \frac{2}{(1+z)^2} & 0 \leq z \leq 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Q17.

$$\begin{aligned}\text{RHS} &= \mathbb{E}[\text{var}(X/Y)] + \text{var}(\mathbb{E}[X/Y]) \\ &= \mathbb{E}\left[\mathbb{E}[X^2/Y] - (\mathbb{E}[X/Y])^2\right] \ + \ \left(\mathbb{E}\left[(\mathbb{E}[X/Y])^2\right] - (\mathbb{E}[\mathbb{E}[X/Y]])^2\right) \\ &= \mathbb{E}\left[\mathbb{E}[X^2/Y]\right] - \mathbb{E}\left[(\mathbb{E}[X/Y])^2\right] \ + \ \mathbb{E}\left[(\mathbb{E}[X/Y])^2\right] - (\mathbb{E}[X])^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{var}(X).\end{aligned}$$

Q18. It can be shown that

$$\left(\mathbb{E}\left[|X|^m\right]\right)^{\frac{1}{m}} \le \left(\mathbb{E}\left[|X|^n\right]\right)^{\frac{1}{n}} \quad \text{for } 1 < m < n$$

by applying Jensen's inequality by letting g to be the function $g(x) = x^{\frac{n}{m}}$ (convince yourself that g is convex) over the random variable $|X|^m$.
Hence we have

$$\left(\mathbb{E}\left[|X_n - X|^s\right]\right)^{\frac{1}{s}} \le \left(\mathbb{E}\left[|X_n - X|^r\right]\right)^{\frac{1}{r}}$$

Here RHS goes to 0 as $n \to \infty$. LHS is a non-negative sequence upper-bounded by RHS. So by sandwitching, LHS also goes to 0 as $n \to \infty$. Therefore $\{X_n\} \xrightarrow{s} X$.

Q19. We will show that the sequence of mgf's of U_n converge (pointwise) to mgf of U. Then this would say $\{U_n\} \xrightarrow{\mathcal{D}} U$.

$$\begin{aligned}M_{U_n}(s) &= \mathbb{E}\left[e^{sU_n}\right] \\ &= \mathbb{E}\left[e^{s\sum_{i=1}^{n}\frac{X_i}{10^i}}\right] \\ &= \prod_{i=1}^{n}\mathbb{E}\left[e^{s\frac{X_i}{10^i}}\right] \qquad \text{[since } X_i\text{'s are independent]}\end{aligned}$$

Now,

$$\begin{aligned}M_{X_i}(s) &= \sum_{j=0}^{9} e^{sj}\frac{1}{10} \\ &= \frac{1}{10}\left[1 + e^s + e^{2s} + \cdots + e^{9s}\right] \\ &= \frac{e^{10s} - 1}{10(e^s - 1)}\end{aligned}$$

$$\begin{aligned}\Rightarrow M_{U_n}(s) &= \frac{e^s - 1}{10(e^{s/10} - 1)} \times \frac{e^{s/10} - 1}{10(e^{s/100} - 1)} \times \cdots \frac{e^{s/10^{n-1}} - 1}{10(e^{s/10^n} - 1)} \\ &= \frac{e^s - 1}{10^n(e^{s/10^n} - 1)} \qquad \text{[canceling consecutive numerator and denominator]} \\ &= \frac{e^s - 1}{10^n\left(\frac{s}{10^n} + \frac{s^2}{2!\,10^{2n}} + \cdots\right)} \\ &= \frac{e^s - 1}{\left(s + \frac{s^2}{2!\,10^n} + \frac{s^3}{3!\,10^{2n}} + \cdots\right)}\end{aligned}$$

$\Rightarrow \lim_{n\to\infty} M_{U_n}(s) = (e^s - 1)/s$, which is indeed the mgf of $U[0, 1]$.

Q20. 40 coins are flipped. Probability of head $p = 0.5$. Let $X_1, X_2, \dots X_{40}$ be the Bernoulli RVs. We have,

$$S_n = X_1 + X_2 + \cdots X_{40}.$$

Hence

$$\mathbb{E}[S_n] = 40 \cdot \mathbb{E}[X_i] = 40 \times 0.5 = 20$$

and

$$\text{var}(S_n) = 40 \cdot \text{var}(X_i) = 40 \times 0.5 \times 0.5 = 10.$$

According to CLT,

$$\frac{S_n - 20}{\sqrt{10}} \sim N(0,1).$$

We want to find $P[S_n = 20]$. Here S_n is discrete RV while $N(0,1)$ is continous. Hence to approximate using CLT, we find the probability of S_n taking values between 19.5 and 20.5.

$$\begin{aligned} P[S_n = 20] &= P[19.5 < S_n \leq 20.5] \\ &= P\left[\frac{19.5 - 20}{\sqrt{10}} < \tilde{S_n} \leq \frac{20.5 - 20}{\sqrt{10}}\right] \\ \text{(by CLT)} &\approx \Phi(0.16) - \Phi(-0.16) \\ &= 2\Phi(0.16) - 1 = 0.1272. \end{aligned}$$

The exact value is given by binomial distribution, i.e. $P[S_n = 20] = {}^{40}C_{20}(0.5)^{40} = 0.1268$.
These values are very close, showing that even with 40 iid RVs, CLT gives good approximation of $\tilde{S_n}$.

Q21.

$$F_{\tilde{S_n}}(y) \approx \Phi\left(\frac{(y-\mu)\sqrt{n}}{\sigma}\right)$$

Given $\mu = 167$, $\sigma = 27$.
For $n = 36$,

$$\begin{aligned} P[163 < \tilde{S_n} \leq 170] &= F_{\tilde{S_n}}(170) - F_{\tilde{S_n}}(163) \\ &\approx \Phi(0.66) - \Phi(-0.88) \\ &= 0.7475 - 0.1870 = 0.5605 \end{aligned}$$

Similarly, for $n = 144$,

$$\begin{aligned} P[163 < \tilde{S_n} \leq 170] &= F_{\tilde{S_n}}(170) - F_{\tilde{S_n}}(163) \\ &\approx \Phi(1.33) - \Phi(-1.77) \\ &= 0.9088 - 0.0377 = 0.8711 \end{aligned}$$

Q22. We have $Y = AX$ where $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 0 & 5 \end{bmatrix}$.
X is (multivariate) standard normal, $\mu_X = 0$ and $\Sigma_X = I$. Since matrix A

has linearly independent rows, Y will be multivariate normal RV with $\mu_Y = 0$, $\Sigma_Y = A\Sigma_X A^\top = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 35 & 11 \\ 11 & 29 \end{bmatrix}$.

$$\Rightarrow \quad f_Y(\bar{y}) = \frac{1}{2\pi\left(35 \times 29 - 11^2\right)^{1/2}} \; e^{\bar{y}^\top \Sigma_Y^{-1} \bar{y}}.$$

(23) $$S_n = \frac{\sum_{i=1}^{n}\left(X_i - \frac{\sum_{j=1}^{n} X_j}{n}\right)^2}{n}$$

$$= \frac{\sum_{i=1}^{n}\left(X_i^2 - 2X_i\left(\sum_{j=1}^{n} X_j/n\right) + \left(\sum_{j=1}^{n} X_j\right)^2/n^2\right)}{n}$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - \left(\frac{\sum_{j=1}^{n} X_j}{n}\right)^2 \quad \text{— (1)}$$

First term → $E[X_i^2]$ a.s. as $n\to\infty$; second term → $(E[X_i])^2$ a.s. as $n\to\infty$ (by strong law of large numbers)

$\Rightarrow \{S_n\} \longrightarrow E[X_i^2] - (E[X_i])^2 = \text{var}(X_i)$

$\therefore$ S_n is an estimator of var (X_i)

Also by (1), $$E[S_n] = E[X_i^2] - \frac{E\left[\left(\sum_{j=1}^{n} X_j\right)^2\right]}{n^2}$$

$$= E[X_i^2] - \frac{\sum_{j=1}^{n} E[X_j^2] - \sum_{i\neq j} E[X_i]E[X_j]}{n^2}$$

$$= E[X_i^2] - \frac{n E[X_i^2] - n(n-1)(E[X_i])^2}{n^2}$$

$$= \frac{n-1}{n}\,\text{var}(X_i)$$

$\therefore$ It is not an unbiased estimator.

~~(24) $E[X^T \Sigma^{-1} X]$~~

(24)

$$E[X^T \Sigma^{-1} X] = E[\text{trace}(X^T \Sigma^{-1} X)]$$

→ trace of number is number

$$= E[\text{trace}(\Sigma^{-1} X X^T)]$$

← trace is invariant under cyclic permutations

$$= \text{trace}(E[\Sigma^{-1} X X^T])$$

← linearity of expectation

$$= \text{trace}(\Sigma^{-1} E[X X^T])$$

$$= \text{trace}(\Sigma^{-1} \Sigma) = \text{trace}(I) = n$$