CS 747, Autumn 2022: Lecture 9

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Markov Decision Problems

1. Review of linear programming
2. MDP planning through linear programming

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Linear Programming

- To solve for real-valued variables $x_1, x_2, \ldots, x_m$ such that
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subject to: //Constraints

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x_1 + x_2 &\leq 9, && \text{(C1)}\\
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- Today's solvers (commercial, as well as open source) can handle LPs with millions of variables.

Conceptual Steps towards Solving a Linear Program

- **Step 1**: Identify the feasible set, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.

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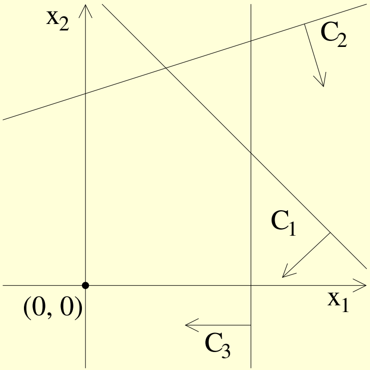

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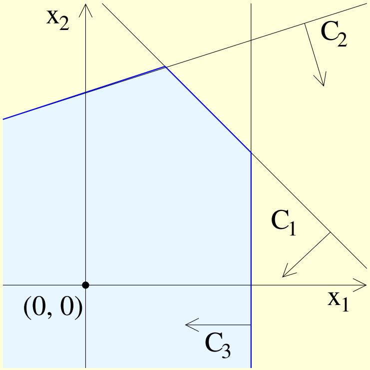

Conceptual Steps towards Solving a Linear Program

- **Step 1**: Identify the feasible set, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.
- **Step 2**: Identify points within the feasible set that maximise the objective. Usually a single point.

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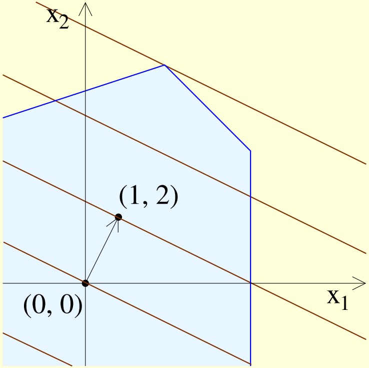

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- Engineer's focus is on formulating, rather than solving, LP.

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Bellman Optimality Equations as an LP

- Bellman optimality equations: for $s \in S$,

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Can we construct an objective function for which $V^\star$ is the sole optimiser?

Vector Comparison

- For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ (equivalently $X, Y \in \mathbb{R}^n$), we define

$$X \succeq Y \iff \forall s \in S : X(s) \geq Y(s),$$
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- Also note that if $\pi_1 \succeq \pi_2$ and $\pi_2 \succeq \pi_1$, then $V^{\pi_1} = V^{\pi_2}$.

$B^\star$ Preserves $\succeq$

- **Fact.** For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$,
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$$(B^\star(X))(s) - (B^\star(Y))(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma X(s')\} -$$
$$\max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma Y(s')\}$$
$$\geq \gamma \min_{a \in A} \sum_{s' \in S} T(s, a, s')\{X(s') - Y(s')\} \geq 0.$$

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- We "linearise" this result: for $V : S \to R$ in the feasible set.

$$\textstyle\sum_{s \in S} V(s) \geq \sum_{s \in S} V^\star(s).$$

Linear Programming Formulation

$$\text{Maximise} \left(-\sum_{s \in S} V(s) \right)$$

subject to

$$V(s) \geq \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma V(s')\}, \forall s \in S, a \in A.$$

- This LP has n variables, nk constraints.

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- There is also a *dual* LP formulation with nk variables and n constraints. See Littman et al. (1995) if interested.

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Next class: policy iteration.