CS 747, Autumn 2022: Lecture 13

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Autumn 2022

Reinforcement Learning

1. Reinforcement learning problem: prediction and control
2. Some natural assumptions
3. Basic algorithm for control

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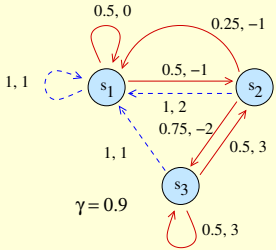
Agent-Environment Interaction

Underlying MDP:



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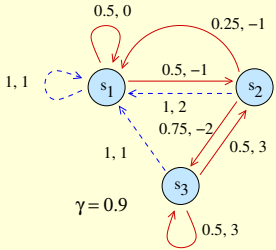

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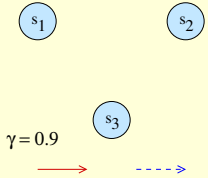

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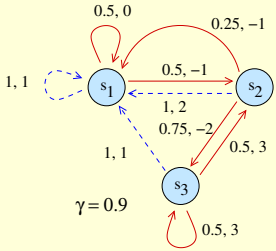

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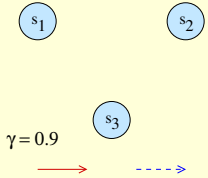

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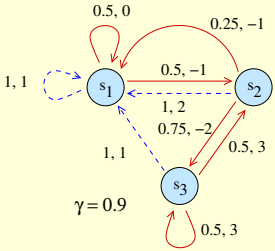

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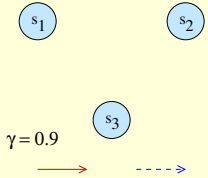

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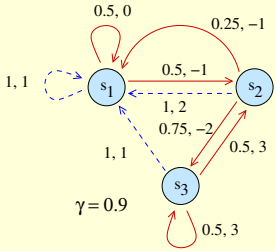

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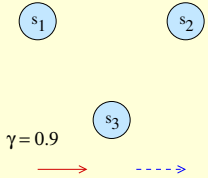

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- Possible history: $s_2, \texttt{RED}, -2, s_3, \texttt{BLUE}, 1, s_1, \texttt{RED}, 0, s_1, \ldots$.

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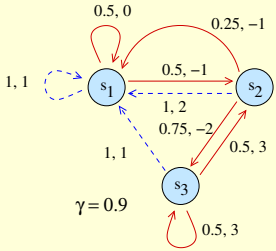

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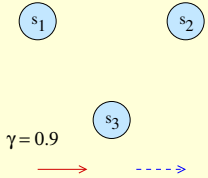

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- Environment (MDP) decides next state and reward.
- Possible history: $s_2, \texttt{RED}, -2, s_3, \texttt{BLUE}, 1, s_1, \texttt{RED}, 0, s_1, \ldots$.
- History conveys information about the MDP to the agent.

The Control Problem

- For $t \geq 0$, let $h^t = (s^0, a^0, r^0, s^1, a^1, r^1, s^2, \ldots, s^t)$ denote a t-length history.

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- Actions are selected by the learning algorithm (agent);
  next states and rewards by the MDP (environment).
- **Control problem**: Can we construct L such that

$$\lim_{H \to \infty} \frac{1}{H} \left(\sum_{t=0}^{H-1} \mathbb{P}\{a^t \sim L(h^t) \text{ is an optimal action for } s^t\} \right) = 1?$$

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- **Prediction problem**: Can we construct L such that

$$\lim_{t \to \infty} \hat{V}^t = V^\pi?$$

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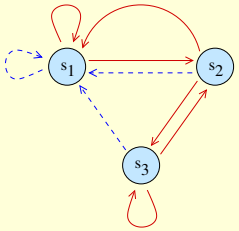
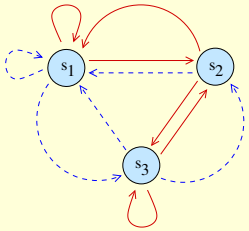
Assumption 1: Irreducibility

- Fix an MDP $M = (S, A, T, R, \gamma)$ and a policy π.
- Draw a graph with states as vertices and every non-zero-probability transition under π as a directed edge.
- Is there a directed path from s to s' for every $s, s' \in S$?
- If yes, M is irreducible under π.
- If M is irreducible under all $\pi \in \Pi$, then M is irreducible.



Reducible



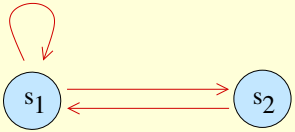
Irreducible

Assumption 2: Aperiodicity

- Fix an MDP $M = (S, A, T, R, \gamma)$ and a policy π.
- For $s \in S$, $t \geq 1$, let $X(s, t)$ be the set of all states s' s. t. there is a non-zero probability of reaching s' in exactly t steps by starting at s and following π.
- For $s \in S$, let $Y(s)$ be the set of all $t \geq 1$ such that $s \in X(s, t)$; let $p(s) = \gcd(Y(s))$.
- M is aperiodic under π if for all $s \in S$: $p(s) = 1$.
- If M is aperiodic under all $\pi \in \Pi$, then M is aperiodic.

$Y(s_1) = \{2, 4, 6, \dots\}$.

Periodic.



$Y(s_1) = \{1, 2, 3, \dots, \}$.

$Y(s_2) = \{2, 3, 4, \dots, \}$.

Aperiodic.

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- We'll use ergodicity in some of the later lectures.

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A Model-based Approach

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- We must visit every state-action pair infinitely often.
- Remember GLIE?

Algorithm

Model-based RL

//Initialisation
For $s, s' \in S, a \in A$:
    $\hat{T}[s][a][s'] \leftarrow 0$; $\hat{R}[s][a][s'] \leftarrow 0$.
For $s, s' \in S, a \in A$:
    $totalTransitions[s][a][s'] \leftarrow 0$;
    $totalReward[s][a][s'] \leftarrow 0$.
For $s \in S, a \in A$:
    $totalVisits[s][a] \leftarrow 0$.
$modelValid \leftarrow False$.

Assume that the agent is born in state s^0. //Continued on next slide.

Algorithm

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//For ever
For $t = 0, 1, 2, \ldots$:
    If *modelValid*:
        $\pi^{opt} \leftarrow MDPPlan(S, A, \hat{T}, \hat{R}, \gamma)$.
        $a^t \leftarrow \begin{cases} \pi^{opt}(s^t) & \text{w. p. } 1 - \epsilon_t, \\ UniformRandom(A) & \text{w. p. } \epsilon_t. \end{cases}$
    Else:
        $a^t \leftarrow UniformRandom(A)$.

    Take action a^t; obtain reward r^t, next state s^{t+1}.
    $UpdateModel(s^t, a^t, r^t, s^{t+1})$.

Algorithm

UpdateModel$(\mathbf{s}, \mathbf{a}, \mathbf{r}, \mathbf{s}')$

$totalTransitions[s][a][s'] \leftarrow totalTransitions[s][a][s'] + 1.$
$totalReward[s][a][s'] \leftarrow totalReward[s][a][s'] + r.$
$totalVisits[s][a] \leftarrow totalVisits[s][a] + 1.$

For $s'' \in S$:
$\hat{T}[s][a][s''] \leftarrow \frac{totalTransitions[s][a][s'']}{totalVisits[s][a]}.$

$\hat{R}[s][a][s'] \leftarrow \frac{totalReward[s][a][s']}{totalTransitions[s][a][s']}.$

If $\neg modelValid$:
If $\forall s'' \in S, \forall a'' \in A : totalVisits[s''][a''] \geq 1$:
$modelValid \leftarrow True.$

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  Needs irreducibility, not aperiodicity.
- Why is this a “model-based” algorithm?

  Uses $\theta(|S|^2|A|)$ memory. Will soon see a “model-free” method that needs $\theta(|S||A|)$ memory.

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Next week: some approaches for prediction.