CS 747, Autumn 2023: Lecture 9

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Markov Decision Problems

1. Review of linear programming
2. MDP planning through linear programming

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Linear Programming

- To solve for real-valued variables $x_1, x_2, \dots, x_m$ such that
  - a given linear function of the variables is maximised, while
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Maximise $x_1 + 2x_2$ //Objective function
subject to: //Constraints

$$x_1 + x_2 \leq 9, \quad \text{(C1)}$$
$$4x_1 - 13x_2 \leq -75, \quad \text{(C2)}$$
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- Well-studied problem with wide-ranging applications in mathematics, engineering.
- Today's solvers (commercial, as well as open source) can handle LPs with millions of variables.

Conceptual Steps towards Solving a Linear Program

- **Step 1**: Identify the feasible set, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.

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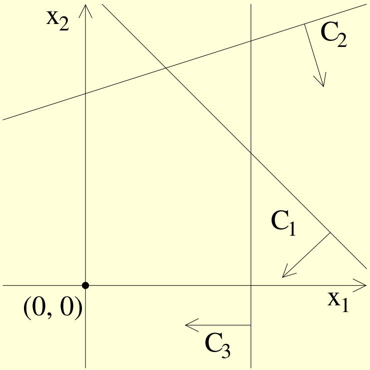

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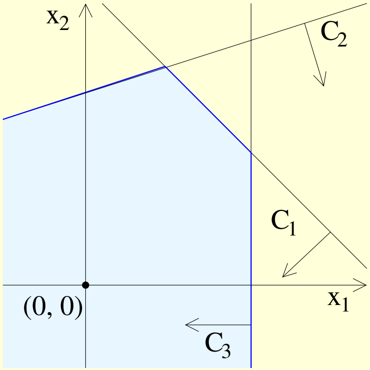

Conceptual Steps towards Solving a Linear Program

- **Step 1**: Identify the feasible set, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.
- **Step 2**: Identify points within the feasible set that maximise the objective. Usually a single point.

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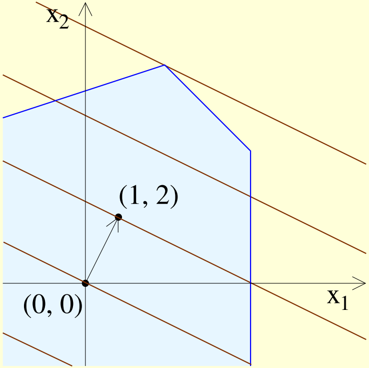

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- Most engineers' focus is on formulating, rather than solving, LP.

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Bellman Optimality Equations as an LP

- Bellman optimality equations: for $s \in S$,

$$V^*(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s') \{R(s, a, s') + \gamma V^*(s')\}.$$

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Can we construct an objective function for which $V^\star$ is the sole optimiser?

Vector Comparison

- For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ (equivalently $X, Y \in \mathbb{R}^n$), we define

$$X \succeq Y \iff \forall s \in S : X(s) \geq Y(s),$$
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- Also note that if $\pi_1 \succeq \pi_2$ and $\pi_2 \succeq \pi_1$, then $V^{\pi_1} = V^{\pi_2}$.

$B^\star$ Preserves $\succeq$

- **Fact.** For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$,

$$X \succeq Y \implies B^\star(X) \succeq B^\star(Y).$$

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$$(B^\star(X))(s) - (B^\star(Y))(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma X(s')\} -$$
$$\max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma Y(s')\}$$
$$\geq \gamma \min_{a \in A} \sum_{s' \in S} T(s, a, s')\{X(s') - Y(s')\} \geq 0.$$

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- For all $V \neq V^\star$ in the feasible set, $V \succ V^\star$. By implication:

$$\sum_{s \in S} V(s) > \sum_{s \in S} V^\star(s).$$

Linear Programming Formulation

Maximise $\left(-\sum_{s \in S} V(s)\right)$

subject to

$$V(s) \geq \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma V(s')\}, \forall s \in S, a \in A.$$

- This LP has n variables, nk constraints.

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Next class: policy iteration.