CS 747, Autumn 2023: Lecture 10

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Autumn 2023

Markov Decision Problems

1. Action value function
2. Policy iteration
   - Policy improvement
   - Policy improvement theorem and proof
   - Policy iteration algorithm
3. History-dependent and stochastic policies

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Action Value Function

- For $\pi \in \Pi, s \in S, a \in A$:

$$Q^{\pi}(s, a) \stackrel{\text{def}}{=} \mathbb{E}[r^0 + \gamma r^1 + \gamma^2 r^2 + \ldots | s^0 = s; a^0 = a; a^t = \pi(s^t) \text{ for } t \geq 1].$$

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- All optimal policies have the same (optimal) action value function $Q^{\star}$.

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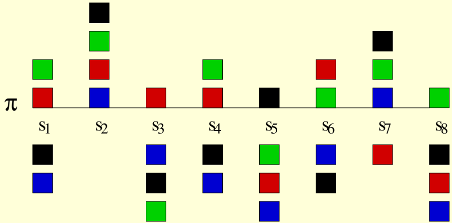

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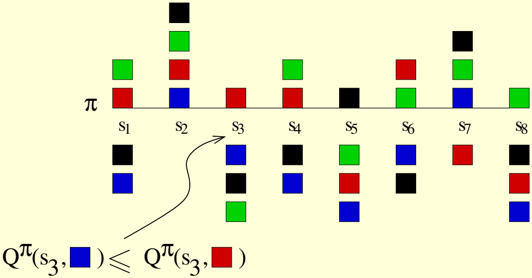

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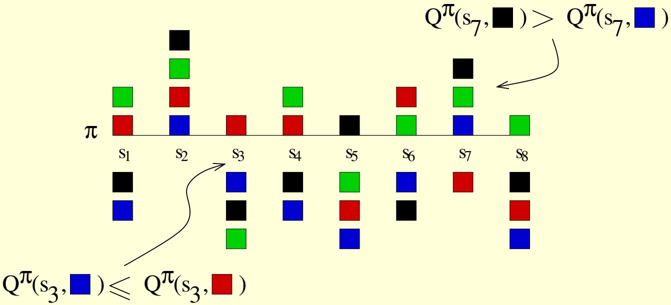

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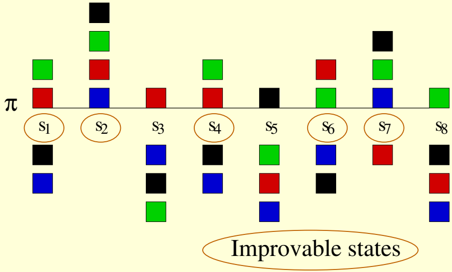

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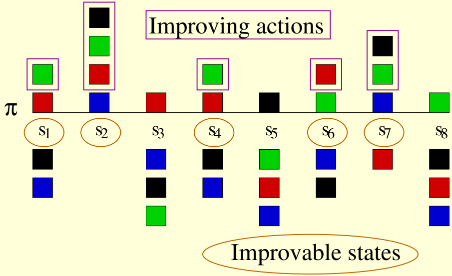

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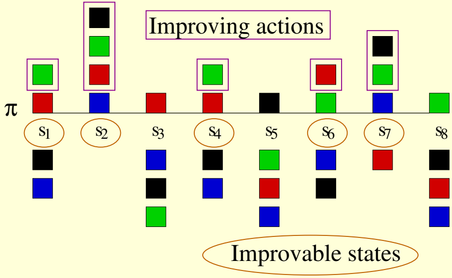

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- Pick one or more improvable states, and in these states,
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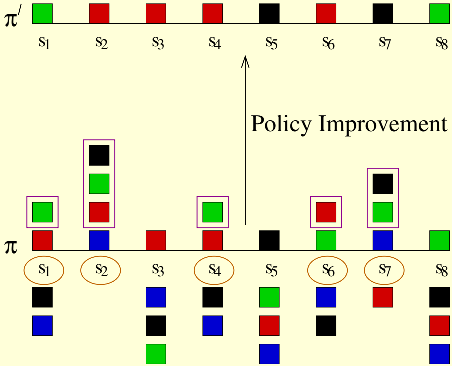

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- Observe that $\textbf{IS}(\pi^\star) = \emptyset \iff B^\star(V^{\pi^\star}) = V^{\pi^\star}$.
- In other words, $V^{\pi^\star}$ satisfies the Bellman optimality equations—which we know has a unique solution. It is a convention to denote $V^{\pi^\star}$ by $V^\star$.

Bellman Operator B^π

- For $\pi \in \Pi$, we define $B^\pi : \mathbb{R}^n \to \mathbb{R}^n$ as follows.
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- Observe that for $\pi, \pi' \in \Pi, \forall s \in S$: $B^{\pi'}(V^\pi)(s) = Q^\pi(s, \pi'(s))$.

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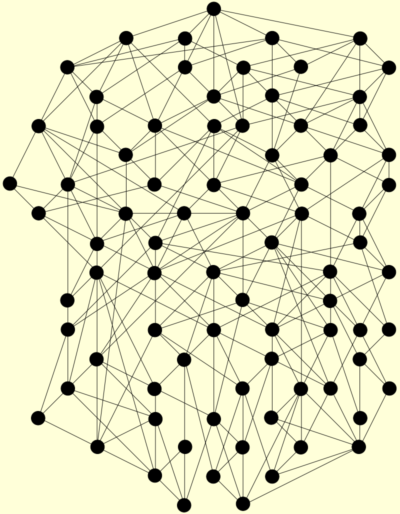

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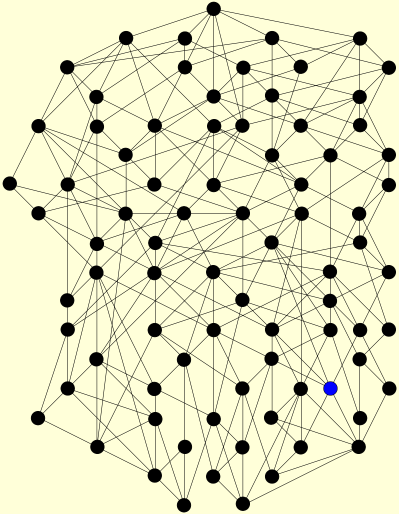

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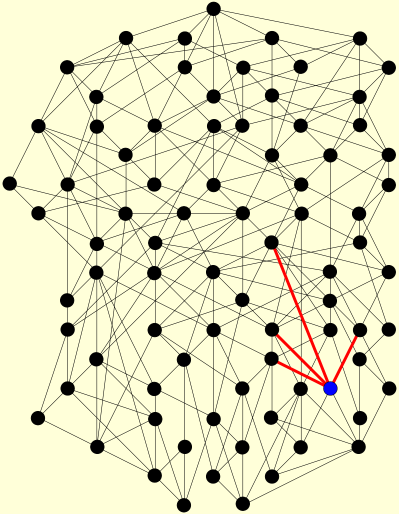

Policy Iteration Algorithm

$\pi \leftarrow$ Arbitrary policy.
While π has improvable states:
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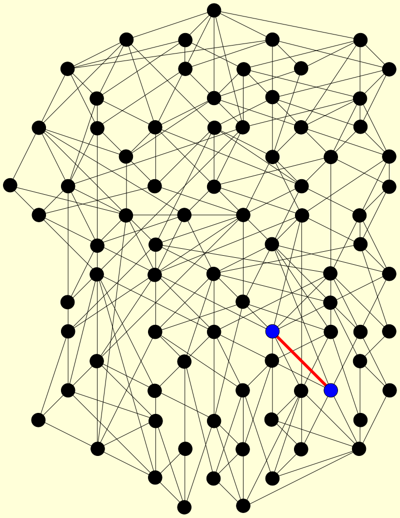

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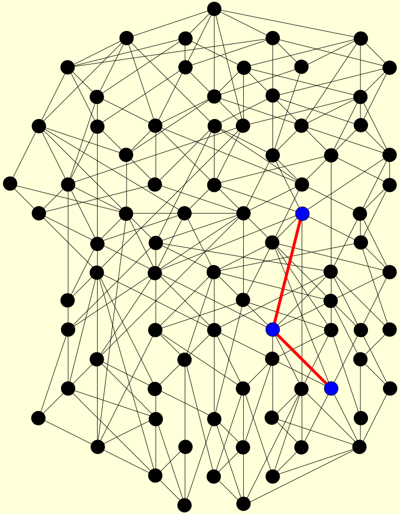

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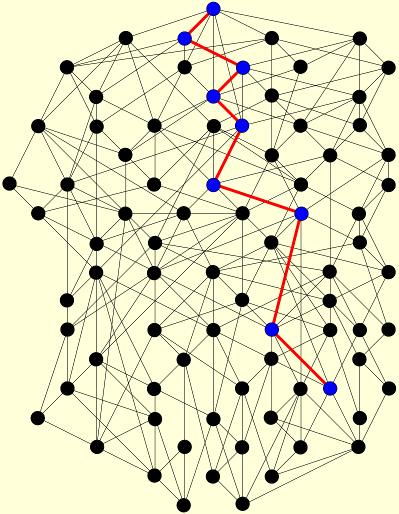

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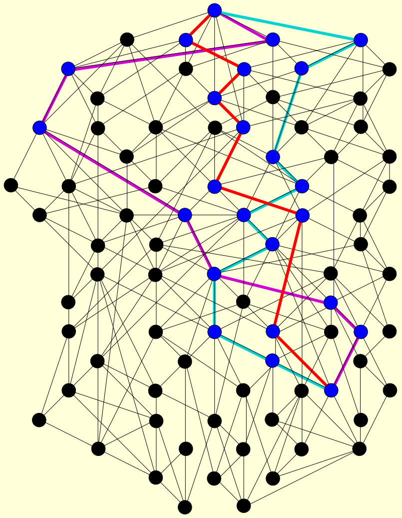
Path taken (and hence the number of iterations) in general depends on the switching strategy.



Markov Decision Problems

1. Action value function
2. Policy iteration
   - Policy improvement
   - Policy improvement theorem and proof
   - Policy iteration algorithm
3. History-dependent and stochastic policies

A More General Class of Policies

- In principle, an agent can follow a policy λ that maps every possible history $s^0, a^0, r^0, s^1, a^1, r^1, \ldots, s^t$ for $t \geq 0$ to a probability distribution over A.
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- Optimal policies for the finite horizon reward setting are in general non-stationary (time-dependent).
- Optimal policies (“strategies”) in many types of multi-player games are in general stochastic (“mixed”) because the next state depends on all the players’ actions, but each player chooses only their own.

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Next class: Running time of policy iteration, review of MDP planning.