CS 747, Autumn 2023: Lecture 13

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Autumn 2023

Reinforcement Learning

1. Reinforcement learning problem: prediction and control
2. Some natural assumptions
3. Basic algorithm for control

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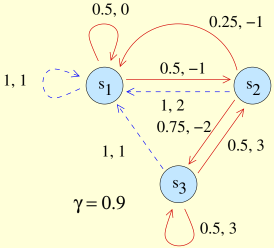
Agent-Environment Interaction

Underlying MDP:



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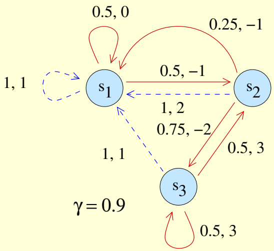
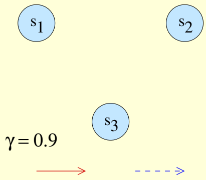


Agent's view:



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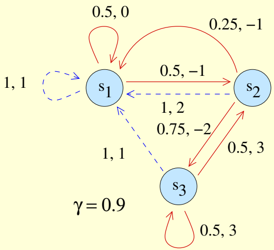

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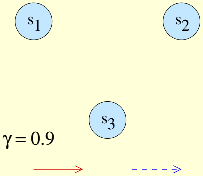

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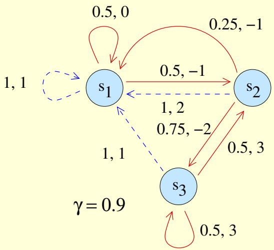

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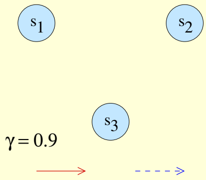

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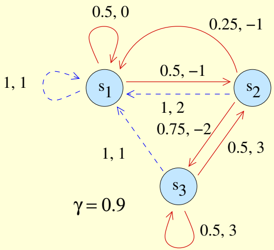

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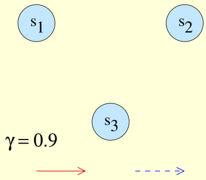

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- Possible history: s_2, RED, -2, s_3, BLUE, 1, s_1, RED, 0, s_1, ...

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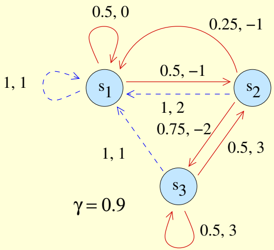

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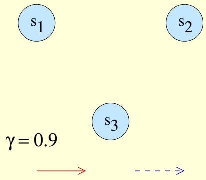

- From current state, agent takes action.
- Environment (MDP) decides next state and reward.
- Possible history: s_2, RED, -2, s_3, BLUE, 1, s_1, RED, 0, s_1, ...
- History conveys information about the MDP to the agent.

The Control Problem

- For $t \geq 0$, let $h^t = (s^0, a^0, r^0, s^1, a^1, r^1, s^2, \ldots, s^t)$ denote a t-length history.

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- Actions are selected by the learning algorithm (agent);
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- **Control problem**: Can we construct L such that

$$\lim_{H \to \infty} \frac{1}{H} \left(\sum_{t=0}^{H-1} \mathbb{P}\{a^t \sim L(h^t) \text{ is an optimal action for } s^t\} \right) = 1?$$

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- **Prediction problem**: Can we construct L such that

$$\lim_{t \to \infty} \hat{V}^t = V^{\pi}?$$

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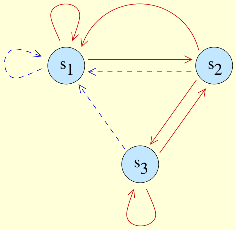
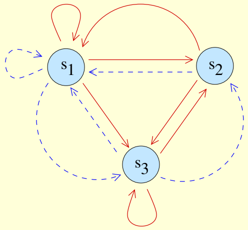
Assumption 1: Irreducibility

- Fix an MDP $M = (S, A, T, R, \gamma)$ and a policy π.
- Draw a graph with states as vertices and every non-zero-probability transition under π as a directed edge.
- Is there a directed path from s to s' for every $s, s' \in S$?
- If yes, M is irreducible under π.
- If M is irreducible under all $\pi \in \Pi$, then M is irreducible.



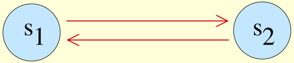
Reducible



Irreducible

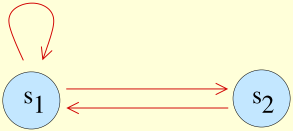
Assumption 2: Aperiodicity

- Fix an MDP $M = (S, A, T, R, \gamma)$ and a policy π.
- For $s \in S$, $t \geq 1$, let $X(s, t)$ be the set of all states s' s. t. there is a non-zero probability of reaching s' in exactly t steps by starting at s and following π.
- For $s \in S$, let $Y(s)$ be the set of all $t \geq 1$ such that $s \in X(s, t)$; let $p(s) = \gcd(Y(s))$.
- M is aperiodic under π if for all $s \in S$: $p(s) = 1$.
- If M is aperiodic under all $\pi \in \Pi$, then M is aperiodic.



$Y(s_1) = \{2, 4, 6, \ldots\}$.

Periodic.



$Y(s_1) = \{1, 2, 3, \ldots, \}$.
$Y(s_2) = \{2, 3, 4, \ldots, \}$.
Aperiodic.

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- We'll use ergodicity in some of the later lectures.

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A Model-based Approach

- A model is an estimate of the MDP, which is usually updated based on experience. We keep estimates $\hat{T}$ and $\hat{R}$, and try to get them to converge to T and R, respectively.

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- We must visit every state-action pair infinitely often.
- Remember GLIE?

Algorithm

Model-based RL

//Initialisation

For $s, s' \in S, a \in A$:

$\hat{T}[s][a][s'] \leftarrow 0; \hat{R}[s][a][s'] \leftarrow 0.$

For $s, s' \in S, a \in A$:

$totalTransitions[s][a][s'] \leftarrow 0;$

$totalReward[s][a][s'] \leftarrow 0.$

For $s \in S, a \in A$:

$totalVisits[s][a] \leftarrow 0.$

$modelValid \leftarrow False.$

Assume that the agent is born in state s^0. //Continued on next slide.

Algorithm

Assume that the agent is born in state s^0. //Continued from previous slide.

//For ever

For $t = 0, 1, 2, \ldots$:

If *modelValid*:

$$\pi^{opt} \leftarrow MDPPlan(S, A, \hat{T}, \hat{R}, \gamma).$$

$$a^t \leftarrow \begin{cases} \pi^{opt}(s^t) & \text{w. p. } 1 - \epsilon_t, \\ UniformRandom(A) & \text{w. p. } \epsilon_t. \end{cases}$$

Else:

$$a^t \leftarrow UniformRandom(A).$$

Take action a^t; obtain reward r^t, next state s^{t+1}.

$UpdateModel(s^t, a^t, r^t, s^{t+1})$.

Algorithm

UpdateModel$(\mathbf{s}, \mathbf{a}, \mathbf{r}, \mathbf{s}')$

$totalTransitions[s][a][s'] \leftarrow totalTransitions[s][a][s'] + 1.$
$totalReward[s][a][s'] \leftarrow totalReward[s][a][s'] + r.$
$totalVisits[s][a] \leftarrow totalVisits[s][a] + 1.$

For $s'' \in S$:

$$\hat{T}[s][a][s''] \leftarrow \frac{totalTransitions[s][a][s'']}{totalVisits[s][a]}.$$

$$\hat{R}[s][a][s'] \leftarrow \frac{totalReward[s][a][s']}{totalTransitions[s][a][s']}.$$

If $\neg modelValid$:

If $\forall s'' \in S, \forall a'' \in A : totalVisits[s''][a''] \geq 1$:

$modelValid \leftarrow True.$

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- Algorithm takes a sub-linear number of sub-optimal actions. Can still be optimised in many ways (computational complexity, exploration, etc.).
- For convergence to optimal behaviour, does the algorithm need irreducibility and aperiodicity?
- Why is this a "model-based" algorithm?

  Uses $\Theta(|S|^2|A|)$ memory. Will soon see a "model-free" method that needs $\Theta(|S||A|)$ memory.

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   Next week: some approaches for prediction.