CS 747, Autumn 2023: Lecture 18

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Autumn 2023

Reinforcement Learning

1. Tile coding

2. Issues in control with function approximation

3. The case for policy search

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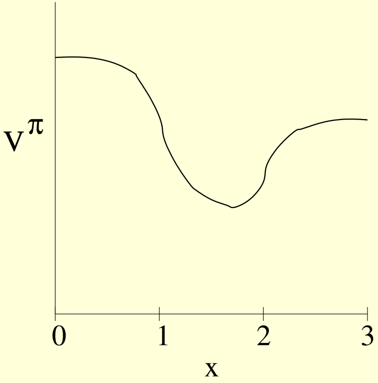
How Good is Linear Function Approximation?



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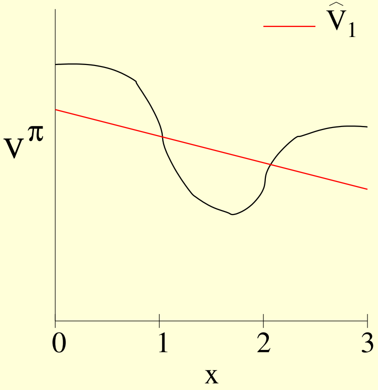

$$\hat{V}_1(x) = w_1 x + w_2.$$

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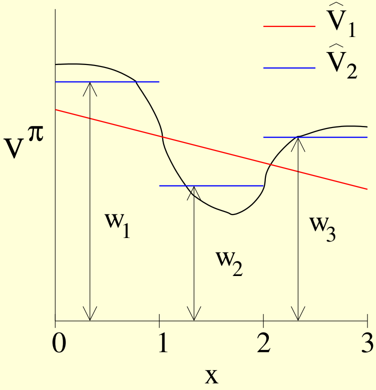

$$b_1 = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$b_2 = \begin{cases} 1 & \text{if } 1 \leq x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$b_3 = \begin{cases} 1 & \text{if } 2 \leq x < 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{V}_2(x) = w_1 b_1 + w_2 b_2 + w_3 b_3.$$

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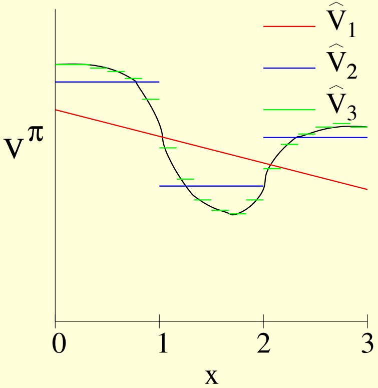

$\hat{V}_3(x)$: 18 piece-wise constants.

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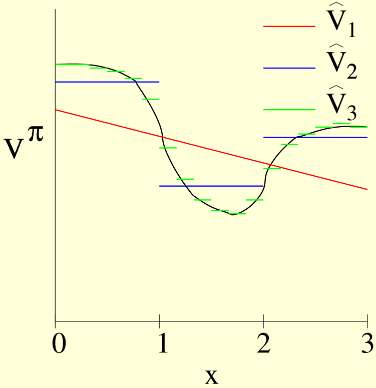

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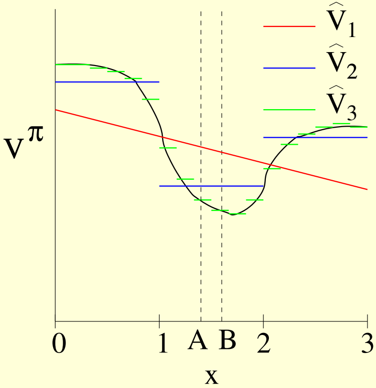

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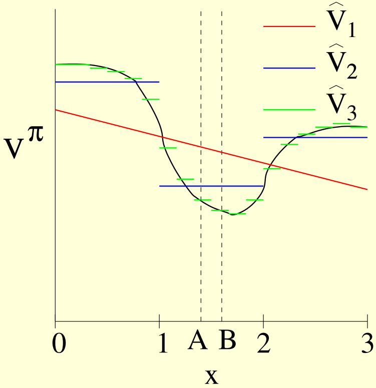
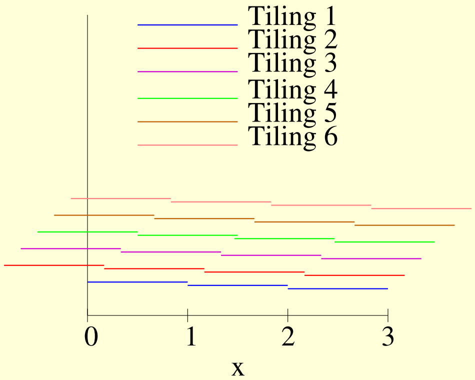


- Is $\hat{V}^3$ the obvious choice?
- $\hat{V}^3$ has the highest resolution, but does not generalise well.
- How to achieve high resolution along with generalisation?

Tile coding



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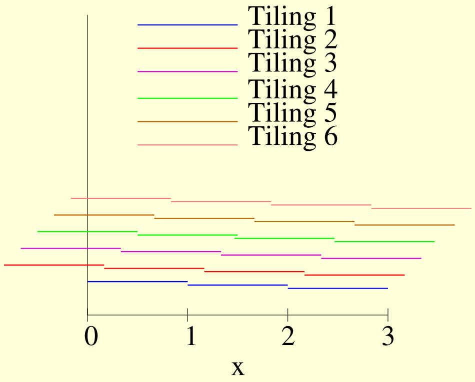

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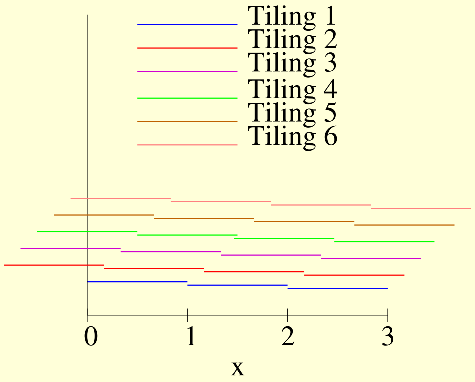

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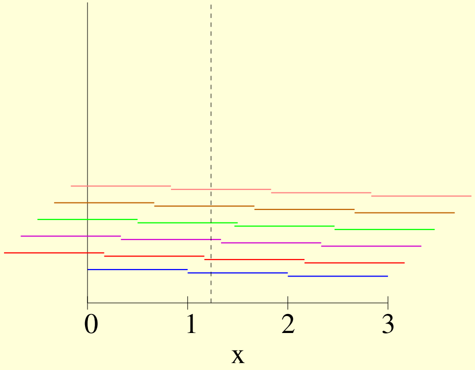

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- The function value of a point is the sum of the weights of the tiles intersecting it (one per tiling).

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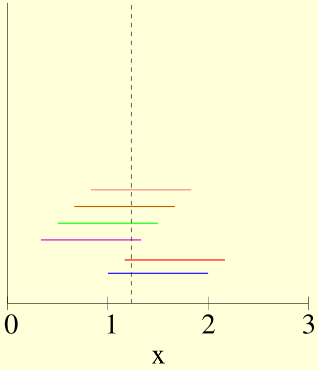

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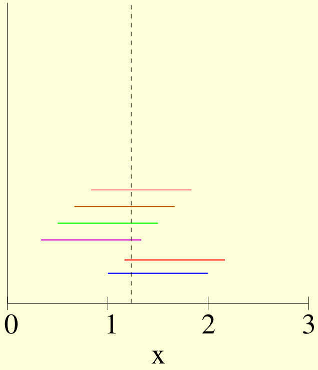

- Each tile is a binary feature.
- Tile width and the number of tilings determine generalisation, resolution.
- Observe that two points more than (tile width / number of tilings) apart can be given arbitrary function values.

Representing $\hat{Q}$

- Given a feature value x as input, the corresponding set of tilings $F : \mathbb{R} \to \mathbb{R}$ returns the sum of the weights of the tiles activated by x.

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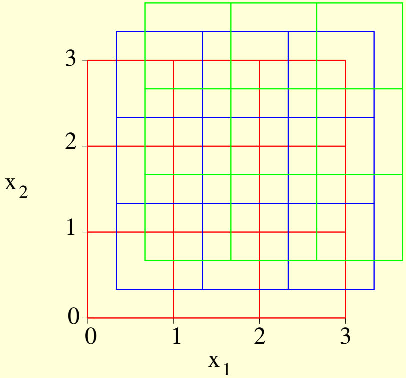
- Usually, tile widths and the number of tilings are configured specifically for each feature. For example, in soccer, could use $2m$ as tile width for "distance" features, and 10° as tile width for "angle" features.

2-*d* Tile coding

- For representing more complex functions, can also have tilings on conjunctions of features (see below for 2 features).



- Introduces more parameters—which could help or hurt.

Tile Coding: Summary

- Linear function approximation does not restrict us to a representation that is linear in the given/raw features.
- Tile coding a standard approach to discretise input features and tune both resolution and generalisation.
- Many empirical successes, especially in conjunction with Linear Sarsa(λ).
- Common to store weights in a hash table (collisions don't seem to hurt much), whose size is set based on practical constraints.
- 1-d tilings most common; rarely see conjunction of 3 or more features.

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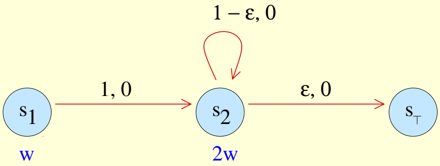
A Counterexample (Tsitsiklis and Van Roy, 1996)



- Prediction problem (policy π).
- Episodic, start state is s_1.
- Observe that $V^{\pi}(s_1) = V^{\pi}(s_2) = 0$.
- Linear function approximation with single parameter w:
  $x(s_1) = 1, x(s_2) = 2$; hence $\hat{V}(s_1) = w, \hat{V}(s_2) = 2w$.

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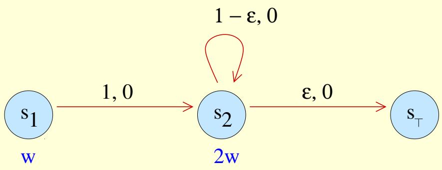

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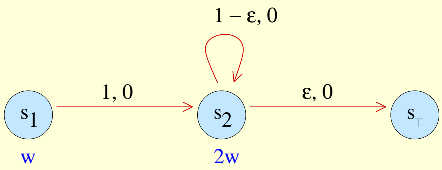

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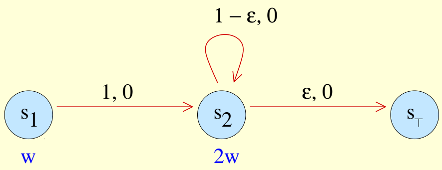

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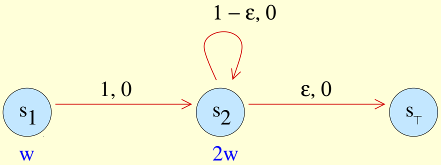
- What's the optimal setting of w?
- $w = 0$ gives the exact answer!
- We design an iteration $w_0 \to w_1 \to w_2 \to \ldots$, and see if it converges to 0.

A Counterexample (Tsitsiklis and Van Roy, 1996)



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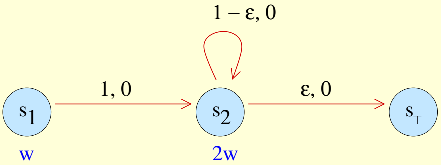

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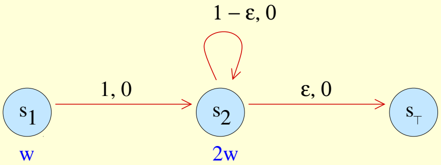

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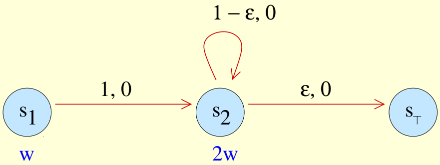

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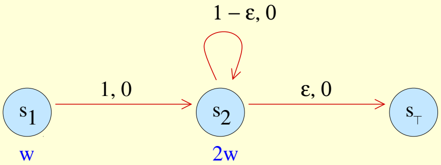

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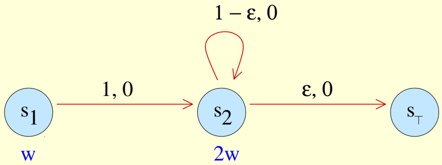

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- Is $\lim_{k \to \infty} w_k = 0$? Let's see.

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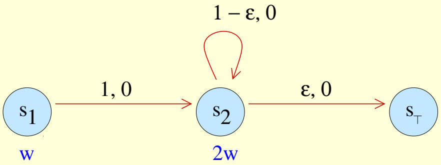

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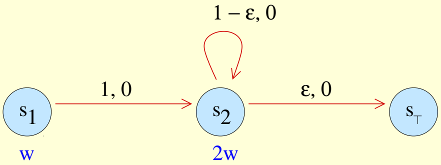

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- For $w_0 = 1$, $\epsilon = 0.1$, $\gamma = 0.99$, $\lim_{k \to \infty} w_k = \infty$; divergence!

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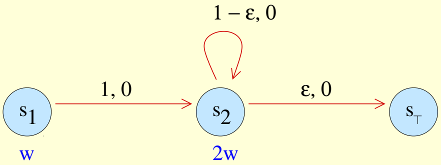

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- The failure owes to the combination of three factors: off-policy updating, generalisation, bootstrapping.

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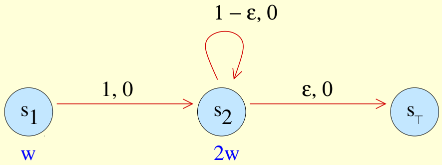

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- The failure owes to the combination of three factors: off-policy updating, generalisation, bootstrapping.
- But these are almost always used together in practice!

Summary of Theoretical Results⋆

Method	Tabular	Linear FA	Non-linear FA
TD(0)	C, O	C	NK
TD(λ), $\lambda \in (0, 1)$	C, O	C	NK
TD(1)	C, O	C, "Best"	C, Local optimum
Sarsa(0)	C, 0	Chattering	NK
Sarsa(λ), $\lambda \in (0, 1)$	NK	Chattering	NK
Sarsa(1)	NK	NK	NK
Q-learning(0)	C, 0	NK	NK

(C: Convergent; O: Optimal; NK: Not known.)

$\star$: to the best of your instructor's knowledge.

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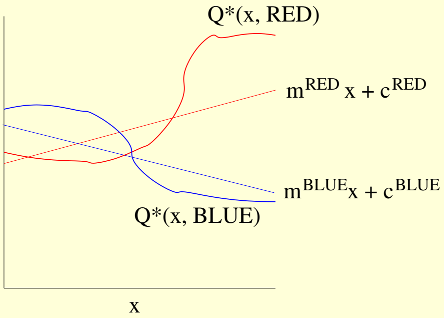
So Near, Yet So Far



- $(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})$ a “good” approximation of $Q^{\star}$.

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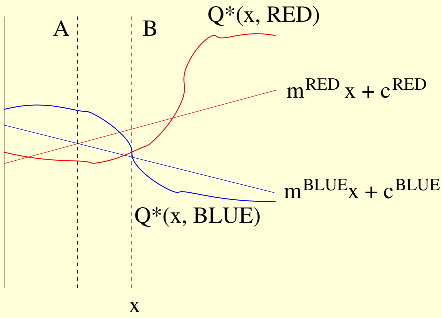

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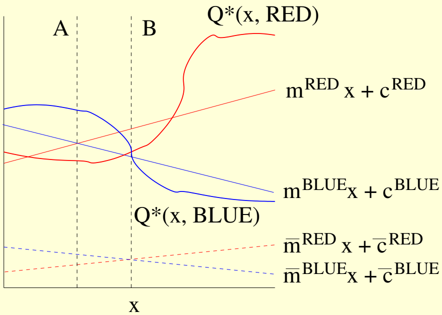

- $(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})$ a “good” approximation of $Q^\star$. But induces non-optimal actions for $x \in (A, B)$.
- $(\bar{m}^{\text{RED}}, \bar{c}^{\text{RED}}, \bar{m}^{\text{BLUE}}, \bar{c}^{\text{BLUE}})$ a “bad” approximation of $Q^\star$. But induces optimal actions for all x!

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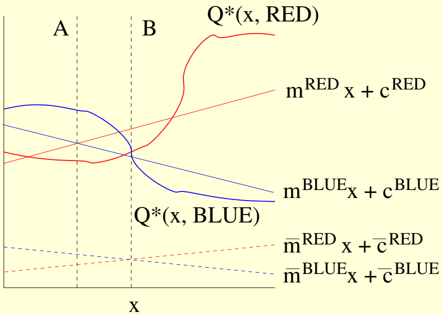

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  But induces optimal actions for all x!
- Perhaps we found $(m^{\mathrm{RED}}, c^{\mathrm{RED}}, m^{\mathrm{BLUE}}, c^{\mathrm{BLUE}})$ by Q-learning.

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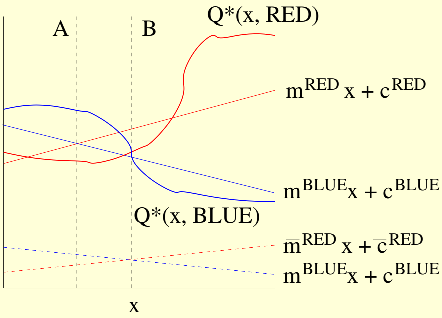

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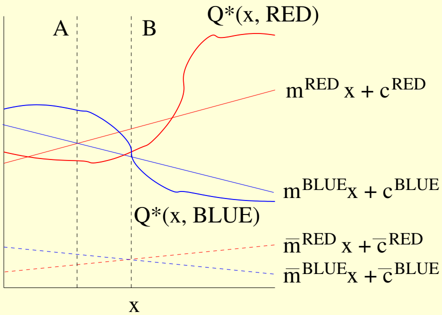
- Perhaps we found $(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})$ by Q-learning.
- How to find $(\bar{m}^{\text{RED}}, \bar{c}^{\text{RED}}, \bar{m}^{\text{BLUE}}, \bar{c}^{\text{BLUE}})$?

So Near, Yet So Far



- $(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})$ a "good" approximation of $Q^{\star}$.
  But induces non-optimal actions for $x \in (A, B)$.
- $(\bar{m}^{\text{RED}}, \bar{c}^{\text{RED}}, \bar{m}^{\text{BLUE}}, \bar{c}^{\text{BLUE}})$ a "bad" approximation of $Q^{\star}$.
  But induces optimal actions for all x!
- Perhaps we found $(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})$ by Q-learning.
- How to find $(\bar{m}^{\text{RED}}, \bar{c}^{\text{RED}}, \bar{m}^{\text{BLUE}}, \bar{c}^{\text{BLUE}})$? **Next week:** policy search.