CS 435 : Linear Optimization Fall 2008

Lecture 23: Matchings in bipartite graphs

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1 Matching in Bipartite Graphs

We now look at matchings from the primal dual perspective. Our objective is an algorithm for finding the maximum weight matching in a bipartite graph. As has been seen earlier, the Primal-Dual algorithm lets us design an algorithm for the weighted case, if we know an algorithm for the unweighted case.

Let us first consider the LP formulation for the unweighted case. The vertex set consists of two independent sets U and V and edges are incident to one vertex in each set.

1.1 The Primal

Variables: One variable for each edge. X_{uv} represents edge $\{u, v\}$. It is one if the edge is picked and zero otherwise. The cost is then **Cost**: $max \sum_{uv} X_{uv}$ What would the constraints be? **Constraints**: For every vertex, there is at most one edge incident on it.

$$\forall u \in U, \sum_{w} X_{uw} \leq 1 \tag{1}$$

$$\forall v \in V, \sum_{p} X_{pv} \leq 1 \tag{2}$$

$$\forall u, v, 0 \leq X_{uv} \leq 1, X_{uv} \text{ is integral} \tag{3}$$

It is instructive to consider the dual, dropping the integrality constraint. We can also, as usual, drop the $X_{uv} \leq 1$ constraint.

1.2 The Dual

For the dual, the number of variables is equal to the number of vertices in the bipartite graph. **Cost**:

Figure 1: Variables in the dual represent the vertices

$min \sum_u Y_u + \sum_v Z_v$. The first term in the sum stands for vertices of set U, while the second term stands for vertices in the set V. **Constraints**: For edge $\{p, q\}$,

$$Y_p + Z_q \geq 1 \tag{4}$$

$$Y_p, Z_q \geq 0 \tag{5}$$

Can we interpret the dual? If this were an integral problem, it looks like we have to pick a minimum number of vertices so that for each edge, one of the vertices must be picked. This is the familiar minimum vertex cover problem!

2 Minimum vertex cover from the maximum matching in bipartite graphs

We know how to find the maximum matching in a bipartite graph. Can we find a minimum vertex cover in polynomial time? The answer is yes. If V is a vertex cover and M is a maximum matching for a bipartite graph, then $|V| \geq |M|$. This is because, any vertex cover has to include at least one end-point of an edge of a maximum matching. In fact, we can say more. *In any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.* This means to find a vertex cover, we can find a maximum matching and then we have to figure out how to put one end-point of each edge such that it forms a vertex cover. Which? Supposing that u is an unmatched vertex, then by the previous discussion all of its matched neighbours have to be in the vertex cover. If we put a vertex in the vertex cover we know that its mate in the matching is not in the vertex cover. Which means all of its neighbours which are matched must be in the vertex cover and so on. Let us formalise this. What if there are no unmatched vertices?

Let $G(U, V, E)$ be a graph and M be a maximum matching for the graph. Let U', V' be the set of those vertices in M which are at odd distance along an alternating path from some unmatched vertex in G. Let M' denote the subset of matched edges neither of whose end-points is selected above. Then $U' \cup V' \cup (M' \cap U)$ is a vertex cover of G. Why?

The edges in the matching can be partitioned into two parts: those in M' and those in $M'' = M \setminus M'$. The two end-points of each edge in M'' can be classified into two parts: one will be at odd distance along alternating paths from an unmatched vertex and the other at even distance. Note that a vertex cannot be at both an odd distance and an even distance along alternating paths from unmatched vertices since that will result in an augmenting path and the matching will not be maximum. For a matching M let $V(M)$ denote the set of end-points of M. We need to show that every edge in the graph has one end-point in the vertex cover. If we consider edges which has one end-point in $V(M')$ and one in $V(M'')$, the end-point in $V(M'')$ will be at an odd distance from an unmatched vertex and hence in the vertex cover. It is easy to see, and left to the reader, to prove that all the other edges will also be covered.

Figure 2: Minimum vertex cover from maximum matching