CS206 Homework #1

Max marks: 85 **Due Feb 18, 2004**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Do not copy solutions from others.***

1. Let p and q be atomic propositions that take values from the set $\{\mathsf{True}, \mathsf{False}\}$. Consider the following two formulae:
   $\phi_1 = (p \rightarrow \neg\phi_2)$
   $\phi_2 = (q \rightarrow \neg\phi_1)$.

   (a) *[10 marks]* Show using natural deduction that $\vdash \phi_1 \vee \phi_2$.

   Note: You may use the Law of Excluded Middle ***at most once*** in your proof. Other than this, you ***must use only the basic introduction and elimination rules*** of natural deduction. In order to score (partial or full) marks, you must also label each step of your proof and show at each step which of the basic "introduction" (e.g., $\rightarrow_i$) or "elimination" (e.g., $\rightarrow_e$) rules are being applied.

   (b) *[10 marks]* Show that there are exactly *two pairs* of propositional logic formulae (ϕ_1, ϕ_2) that satisfy the above definitions. You must also give justification for your answer. Simply giving the pairs of formulae will fetch no marks.

2. Consider the following set of atomic propositions encoding English language declarations:

   - ***Proposition a***: Mr. X is outdoors.
   - ***Proposition b***: It is raining.
   - ***Proposition c***: Mr. X has an umbrella.
   - ***Proposition d***: Mr. X gets wet.
   - ***Proposition e***: Mr. X catches a cold.

   Suppose we are also told the following facts about Mr. X:

   - If Mr. X does not have an umbrella and he does not get wet, then either he is not outdoors or it is not raining or both.
   - If Mr. X is outdoors and has an umbrella, then it never rains (poor Mr. X!).
   - If Mr. X gets wet, he catches a cold.
   - If Mr. X is outdoors and he catches a cold, then he does not have an umbrella.

   Using only the above facts, we wish to use our knowledge of reasoning about propositional logic to infer whether it is possible for Mr. X to catch a cold.

   (a) *[10 marks]* Formulate the above inferencing problem as a satisfiability checking problem for a suitable propositional logic formula. You must clearly state the reasoning behind your formulation. Answers without adequate justification will fetch no marks.

(b) *[10 marks]* Convert the formula obtained in the above subquestion into an equisatisfiable CNF formula by first converting it to an implication-free negation normal form formula, and then applying the labeled CNF procedure. You must show each step of your conversion process to get partial or full marks.

(c) *[15 marks]* Apply the DPLL procedure to check the satisfiability of the formula given above. You must show each step of applying the procedure. In case your formula is satisfiable, you must give a satisfying assignment. Interpret the meaning of your answer in terms of Mr. X's catching a cold.

3. Using the $\wedge_i, \wedge_e, \vee_i, \vee_e, \rightarrow_i, \rightarrow_e, \bot_i$ (also called $\neg_e$), $\bot_e, \neg_i, \neg\neg_e, \neg\neg_i$ rules, Modus Tollens and ***zero applications*** of the Law of Excluded Middle, prove the following sequents. To score marks, each of your proofs must have no more than 10 steps.

   (a) *[5 marks]* $(\phi_1 \vee \phi_2), (\phi_2 \rightarrow \phi_1) \vdash \phi_1$

   (b) *[5 marks]* $\phi_1 \rightarrow (\phi_2 \rightarrow \neg\phi_1) \vdash \phi_2 \rightarrow \neg\phi_1$

   (c) *[5 marks]* $\vdash (\phi_1 \rightarrow (\phi_2 \rightarrow (\phi_1 \rightarrow \phi_2)))$

4. *[15 marks]* As discussed in class, a propositional logic formula can have other propositional logic formulae as subformulae. Implicit in our definition was the absence of recursive definitions, i.e. a formula can't have itself as a subformula. This allowed us to have a tree-like representation of propositional logic formulae, where the internal nodes of the tree are the propositional connectives and the leaves are the propositional variables or constants. Given truth values of all propositional variables, every such propositional logic formula must always evaluate to a value in $\{\mathsf{True}, \mathsf{False}\}$.

   Now suppose we allow recursive definitions of formulae. Let p and q be propositional variables. Construct a pair of distinct propositional logic formulae $\phi_1(p, q)$ and $\phi_2(p, q)$ using only the connective $\rightarrow$ and the functions ϕ_1 and ϕ_2 (possibly used recursively), such that:

   - There exists an assignment of truth values to p and q for which both ϕ_1 and ϕ_2 evaluate to unique values in $\{\mathsf{True}, \mathsf{False}\}$, and
   - There exists another assignment of truth values to p and q for which neither ϕ_1 nor ϕ_2 evaluates to either True or False. *This should illustrate the dangers of allowing recursive definitions.*

   You must give clear justification for your answer in order to score marks. Simply providing two functions will fetch no marks.