CS206 Mid-Semster Examination

Max marks: 60 **Time: 2 hours**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others.***

1. *[5 + 5 marks]* Consider the predicate logic formula $\phi(y) = \forall x(P(x, y) \to \forall x P(x, y))$, with free variable y.
   (a) *[5 marks]* You are given a model $\mathcal{M}$, consisting of $S = \{a, b, c\}$, and $P = \{(a, c), (a, a), (b, c), (b, a), (c, c), (c, a)\}$. If l is an assignment (or environment) defined by $l(y) = c$, determine whether $\mathcal{M} \models_l \forall y\, \phi(y)$. You must indicate your reasoning in order to score marks.
   (b) *[5 marks]* We wish to construct a new model $\mathcal{M}'$ consisting of the same S as in the previous subquestion, but with a possibly new P', such that $\mathcal{M}' \models_l \neg\phi(y)$. Give a suitable P' for this purpose. You must indicate your reasoning in order to score marks.
   Note that you are required to keep both S and l the same as in the previous questions.

2. *5 + 5 + 5 marks]* Using natural deduction, prove the following sequents:
   (a) *[5 marks]* $x \to (y \to z), u \to (z \to w), z \to (x \to u) \vdash y \to (x \to w))$. You must use only the $\to_i$ and $\to_e$ rules for this problem.
   (b) *[5 marks]* $\forall x(Q(x) \wedge \neg P(y)) \vdash \forall x(P(x) \to Q(y))$. You may assume that y is free for x in Q and x is free for y in P. You must use only $\forall_i$ and $\forall_e$ rules along with the basic introduction and elimination rules for natural deduction in propositional logic for this problem.
   (c) *[5 marks]* $(\exists x\, Q(x, y)) \to (\forall y\, P(y)), y = f(x), \forall z\, Q(z, f(x)) \vdash \forall x\, P(f(x))$. You must use only substitution rules of predicate logic, the fact that $\forall x\, \phi \to \exists x\, \phi$ and the basic elimination and introduction rules of natural deduction in propositional logic for this problem.

   For all subproblems above, no rules other than those explicitly allowed may be used. In addition, all your proofs must have 15 or fewer steps.

3. *[5 + 5 + 5 + 10 marks]* A student wishes to find a satisfying assignment for the propositional logic formula $\phi(x, y, z) = \phi_1(x, y, z) \wedge \phi_2(x, y, z)$. Suppose $\phi_1(x, y, z)$ is the Horn formula $(x \wedge y \to z) \wedge (z \to \mathsf{False}) \wedge (x \to \mathsf{True})$, and $\phi_2(x, y, z)$ is the CNF formula $(\neg x \vee \neg z) \wedge (z \vee y \vee x)$. The student proposes to proceed in three different ways to solve this problem:
   (a) A satisfying assignment for ϕ_1 is obtained using the method for satisfiability checking of Horn formulae discussed in class. Formula ϕ_2 is then simplified using the variable assignments thus found, and the assignments for remaining variables, if any, are obtained by applying the DPLL procedure on the simplified formula.
   (b) A satisfying assignment for ϕ_2 is obtained by applying the DPLL procedure. Formula ϕ_1 is then simplified using the variable assignments thus found. The assignments for remaining variables, if any, are obtained by applying the method for satisfiability checking of Horn formulae discussed in class on the simplified formula.
   (c) DPLL procedure is applied on the entire formula ϕ.

   (i) Indicate in each case, whether the student succeeds in finding a satisfying assignment for ϕ. In each case, you must give the (possibly partial) assignment obtained and show the stepwise execution of each algorithm.
   (ii) Other than method (c) above, does method (a) or (b) guarantee a correct answer for arbitrary Horn formulae ϕ_1 and CNF formulae ϕ_2. If so, give an informal justification. Else, give counterexamples for each case.